THE FEBRUARY MEETING IN NEW YORK

The three hundred seventy-sixth meeting of the American Mathematical Society was held at Columbia University on Friday and Saturday, February 21–22, 1941. The attendance was approximately three hundred fifty including the following two hundred thirty-five members of the Society:

C. R. Adams, R. B. Adams, Leonidas Alaoglu, C. B. Allendoerfer, R. L. Anderson, T. W. Anderson, Lawrence Annenberg, L. A. Aroian, K. J. Arrow, F. E. Baker, S. F. Barber, Joshua Barlaz, P. T. Bateman, Felix Bernstein, E. E. Betz, Garrett Birkhoff, Archie Blake, Gertrude Blanch, R. P. Boas, Salomon Bochner, Frank Boehm, H. F. Bohnenblust, Samuel Borofsky, D. G. Bourgin, C. B. Boyer, A. T. Brauer, H. W. Brinkmann, H. C. Brodie, A. A. F. Brown, B. L. Brown, Jewell H. Bushey, J. H. Bushey, S. S. Cairns, A. D. Campbell, D. E. Christie, J. A. Clarkson, R. M. Cohn, T. F. Cope, Byron Cosby, Richard Courant, Ruth Crucet, H. B. Curry, M. D. Darkow, Norman Davids, D. R. Davis, Jesse Douglas, Arnold Dresden, R. F. Dressler, J. E. Eaton, M. L. Elveback, Paul Erdös, C. J. Everett, J. M. Feld, Aaron Fialkow, F. A. Ficken, M. I. Fields, N. J. Fine, W. B. Fite, D. A. Flanders, G. E. Forsythe, Tomlinson Fort, M. C. Foster, R. M. Foster, J. S. Frame, K. O. Friedrichs, T. C. Fry, H. P. Geiringer, Abe Gelbart, B. P. Gill, R. L. Gilman, J. W. Givens, A. M. Gleason, Michael Golomb, M. C. Graustein, C. H. Graves, M. C. Gray, S. L. Greitzer, H. M. Griffin, C. C. Grove, D. W. Hall, F. C. Hall, Marshall Hall, N. A. Hall, P. R. Halmos, Philip Hartman, C. M. Hebbert, M. H. Heins, Edward Helly, J. G. Herriot, Fritz Herzog, L. S. Hill, Einar Hille, Abraham Hillman, T. R. Hollcroft, G. M. Hopper, Harold Hotelling, S. E. Hotelling, E. A. Hoy, R. Y. Hoy, Chung-Tsi Hsu, E. M. Hull, Alexandra Illmer, R. P. Isaacs, Seymour Jablon, Nathan Jacobson, R. A. Johnson, H. A. Jordan, Mark Kac, I. N. Kagno, Shizuo Kakutani, E. R. van Kampen, Irving Kaplansky, Edward Kasner, L. S. Kennison, J. R. Kline, E. R. Kolchin, B. O. Koopman, Arthur Korn, H. N. Laden, Jack Laderman, H. G. Landau, A. W. Landers, M. K. Landers, G. A. Larew, Solomon Lefschetz, Joseph Lehner, B. A. Lengyel, Howard Levi, Madeline Levin, Marie Litzinger, E. R. Lorch, A. N. Lowan, J. C. C. McKinsey, L. A. MacColl, C. C. MacDuffee, G. R. MacLane, H. F. MacNeish, W. G. Madow, Dorothy Maharam, Dorothy Manning, W. P. Mason, Walther Mayer, A. E. Meder, F. H. Miller, H. L. Mintzer, Don Mittleman, E. C. Molina, Elizabeth Monroe, Richard Morris, Nathan Morrison, Marston Morse, Z. I. Mosesson, G. W. Mullins, F. J. Murray, C. A. Nelson, O. E. Neugebauer, C. O. Oakley, E. G. Olds, P. S. Olmstead, Oystein Ore, J. C. Oxtoby, N. G. Parke, Everett Pitcher, Harry Pollard, George Polya, E. L. Post, Walter Prenowitz, R. G. Putnam, H. A. Rademacher, C. F. Rehberg, H. J. Riblet, Moses Richardson, R. G. D. Richardson, J. S. Ripandelli, J. F. Ritt, I. F. Ritter, H. E. Robbins, S. L. Robinson, Benjamin Rosenbaum, Arthur Sard, S. A. Schelkunoff, Peter Scherk, I. J. Schoenberg, Abraham Schwartz, C. H. W. Sedgewick, L. W. Sheridan, Seymour Sherman, W. A. Shewhart, Max Shiffman, L. P. Siceloff, James Singer, L. L. Smail, M. F. Smiley, P. A. Smith, Virgil Snyder, Abraham Spitzbart, Wolfgang Sternberg, F. M. Stewart, G. R. Stibitz, J. J. Stoker, W. C. Strodt, Alvin Sugar, M. M. Sullivan, J. D. Tamarkin, Alfred Tarski, A. H. Taub, J. H. Taylor, B. J. Tepping, J. M. Thomas, R. M. Thrall, E. W. Titt, W. R. Transue, W. J. Trjitzinsky, J. W. Tukey, Oswald Veblen, Abraham Wald,

G. L. Walker, R. J. Walker, R. M. Walter, Louis Weisner, B. A. Welch, M. E. Wells, O. L. Wheeler, A. L. Whiteman, P. M. Whitman, Hassler Whitney, E. P. Wiggin, S. S. Wilks, W. H. Wise, Audrey Wishard, Jacob Wolfowitz, H. A. Wood, Leo Zippin, O. J. Zobel, Antoni Zygmund.

The opening session on Friday afternoon was a symposium on applied mathematics at which Mr. R. M. Foster presided. The symposium consisted chiefly of two addresses, one by Professor J. J. Stoker entitled *Mathematical problems connected with the bending and buckling of elastic plates* and one by Dr. W. A. Shewhart entitled *Mathematical statistics in mass production*. Discussion of the first address was led by Professor K. O. Friedrichs. Dr. Stewart Way was unable to attend and his remarks were read by the presiding officer. The discussion leaders of the second address were Professors Harold Hotelling and S. S. Wilks. Informal discussions of both addresses followed the prepared discussions.

On Saturday morning there was a session for papers in Analysis, Professor J. D. Tamarkin presiding, and another for Algebra, Topology, and Statistics, Professor Jewell Hughes Bushey presiding.

At a general session Saturday afternoon, at which President Marston Morse presided, the members of the Society voted to amend the By-Laws in accordance with the recommendation of the Council. For the convenience of members the By-Laws in revised form are published in full below.

Article I. Officers

Section 1. There shall be a President, three Vice Presidents, a Secretary, four Associate Secretaries, a Treasurer, and a Librarian.

Section 2. It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter.

Section 3. The Librarian shall have charge of arrangements for the exchange of the Society's publications.

Article II. Board of Trustees

Section 1. There shall be a Board of Trustees consisting of five trustees elected by the Society, in accordance with Article V.

Section 2. The function of the Board of Trustees shall be to receive and administer the funds of the Society, to have full legal control of its investments and properties, to make contracts and, in general, to conduct all business affairs of the Society.

Section 3. The Board of Trustees shall have the power to appoint a manager and such assistants and agents as may be necessary or convenient to facilitate the conduct of the affairs of the Society, and to fix the terms and conditions of their employment. The Board may delegate to the officers of the Society duties and powers normally inhering in their respective corporative offices, subject to supervision by the Board. The Board of Trustees may appoint committees to facilitate the conduct of the business of the Society and delegate to such committees such powers as may be necessary or convenient for the proper exercise of those powers. Agents appointed, or members

of committees designated, by the Board of Trustees need not be members of the Board.

Nothing herein contained shall be construed to empower the Board of Trustees to divest itself of responsibility for, or legal control of, the investments, properties and contracts of the Society.

Article III. Editorial Committees

Section 1. There shall be five Editorial Committees—one of four members for the *Bulletin*, one of three members for the *Transactions*, one of three members for the *Mathematical Reviews*, one of three members for the *Colloquium Publications*, and one consisting of three representatives of the Society on the Board of Editors of the *American Journal of Mathematics*.

Article IV. Council

Section 1. The Council shall consist of fifteen members at large, and the following *ex officio* members: the officers of the Society specified in Article I, the members of the Editorial Committees, ex-secretaries who have served as secretaries for ten years or more, and ex-presidents for a period of six years after the completion of their respective presidential terms.

Section 2. The Council shall formulate and administer the scientific policies of the Society and shall act in an advisory capacity to the Board of Trustees.

Section 3. All members of the Council shall be entitled to be present at any meeting of the Council and to take part in the discussion at that meeting.

Section 4. In the absence of the Secretary from any meeting of the Council, one of the Associate Secretaries present may be designated as Acting Secretary for the meeting, either by written authorization of the Secretary, or, failing that, by majority agreement among the Associate Secretaries present.

Section 5. The voting members at any meeting of the Council shall be the following: the members at large of the Council; the President; the Vice Presidents; the Secretary; the Acting Secretary for the meeting, if one is designated in the absence of the Secretary; the Treasurer; and a member of each Editorial Committee chosen annually by the committee, provided that if such voting member is absent at any meeting of the Council an alternate designated by him from among the membership of the committee shall have the power to vote.

Section 6. The method for settling matters before the Council at any meeting shall be by majority vote of the voting members present. If the result of such a vote is challenged at the time by any member of the Council, it shall be the duty of the Secretary—or of the Acting Secretary for the meeting—or of a member present who shall be designated by the Council for that purpose—to prepare a list of the voting members present, on the basis of which the presiding officer shall determine the true vote by a roll call.

Section 7. A quorum for the transaction of business at any meeting of the Council shall consist of five voting members as defined in Section 5 of this Article.

Section 8. Between meetings of the Council, business may be transacted by a mail vote. Voting members for this purpose shall be the members at large of the Council, the President, the Vice Presidents, the Secretary, the Treasurer, and the voting member of each Editorial Committee as designated in Section 5. An affirmative vote on any proposal by mail shall require at least a majority of the voting members, and at least three-quarters of the total number of votes transmitted to the Secretary before the time announced for the closing of the polls. If two or more voting members request

postponement at the time of voting, action on the matter at issue shall be postponed until the next meeting of the Council, unless at the discretion of the Secretary the question is made the subject of a second vote by mail, in connection with which brief statements of reasons, for and against, are circulated.

Article V. Elections and Terms of Officers

Section 1. The term of office shall be two years in the case of the trustees, the President, the Vice Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years in the case of the Librarian and the members of the Editorial Committees. The term of office for members at large of the Council shall be three years, five of the members at large retiring annually. In every case, however, the officials specified in Articles I, II, III, and IV shall continue to serve until their successors have been duly elected and qualified.

Section 2. The President and Vice Presidents shall not be eligible for immediate re-election to their respective offices. A member at large or an *ex officio* member of the Council shall not be eligible for immediate election (or re-election) as a member at large of the Council.

Section 3. Election of officers specified in Article I, trustees, members of Editorial Committees, and members at large of the Council shall be by ballot at the Annual Meeting. An official ballot shall be sent to each member by the Secretary at least one month before the Annual Meeting, and such ballots, if returned to the Secretary in envelopes bearing the name of the voter and received prior to the closing of the polls, shall be counted at the Annual Meeting. Each ballot shall contain one or more names proposed by the Council for each office to be filled, with blank spaces in which the voter may substitute other names. A plurality of all votes cast, whether cast in person or by mail, shall be necessary for election. In case of failure to secure a plurality for any office, the members present at the Annual Meeting shall choose by ballot between the two having the highest number of votes.

Section 4. If the President of the Society die or resign before the expiration of his term of office, the Council, may, with the approval of the trustees, designate one of the Vice Presidents to serve as President until the next Annual Meeting, when the Society shall elect a President for a two-year term. Such vacancies as may occur at any time in the group consisting of the Vice Presidents, the Secretary, the Associate Secretaries, the Treasurer, the Librarian, and the members of the Editorial Committees may be filled by the Council with the approval of the trustees.

Section 5. If any trustee die or resign during his tenure of office, the vacancy thus created shall be filled for his unexpired term by the Board of Trustees.

Section 6. If any member at large of the Council die or resign more than one year before the expiration of his term, the vacancy for the unexpired term shall be filled by the Society at the next Annual Meeting.

Article VI. Election of Members

Section 1. Election of members shall be by vote of the Council.

Section 2. There shall be four classes of members, ordinary, contributing, institutional contributing, and sustaining.

Section 3. A firm, corporation, institution, association or individual interested in the support of mathematics, may be elected to sustaining membership. A sustaining member shall receive the *Bulletin*, the *Transactions*, and *Mathematical Reviews* (see Article IX), and shall have the privilege of nominating one or more persons for election by the Council to ordinary membership in the Society, the number to be determined

by the Council. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

Section 4. A firm, corporation, institution, or association interested in the support of mathematics may be elected to institutional contributing membership. An institutional contributing member shall receive the *Bulletin* or have the privilege, for each twenty-five dollars contributed, of nominating one person to ordinary membership in the Society. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

Section 5. Except in the case of nominees of sustaining members and of institutional contributing members, application for admission to ordinary membership shall be made by the applicant, on a blank provided by the Secretary, and shall be approved by two members of the Society. Such applications shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail), except in the case of members of other societies entering under special action of the Council.

Section 6. An ordinary member may become a contributing member by paying the dues for such membership. (See Article VII, Section 4.)

Section 7. An ordinary member or a contributing member shall receive the *Bulletin*, except in the case of members whose dues are remitted under Article VII, Section 10.

Article VII. Dues

Section 1. Persons elected to ordinary membership in the Society by the Council, under the provisions of Article VI, Section 5, shall be admitted to membership upon the payment, within sixty days of the date of their election, of an initiation fee of five dollars.

Section 2. The annual dues of persons elected by the Council to ordinary membership under the provisions of Article VI, Section 5, shall be eight dollars, with the following exceptions: (1) during the first three years of membership, the annual dues shall be six dollars; a person shall be considered to have completed his first year of membership on January 1 following his election; (2) the amount of dues may be altered by reciprocity agreements with other societies; (3) the Council may make special rulings in exceptional cases, with the approval of the Board of Trustees. Each new member shall pay in proportion to the unexpired fraction of the year at the time of his election.

Section 3. Four dollars and fifty cents of the annual dues of those members who receive the *Bulletin* under the provisions of Article VI shall be allocated in payment therefor.

Section 4. The minimum dues for a contributing member shall be fifteen dollars per year. Members may, upon their own initiative, pay larger dues.

Section 5. The dues for an institutional contributing member for any year shall not be less than twenty-five dollars. Institutions may pay larger dues.

Section 6. The dues of a sustaining member for any year shall not be less than one hundred dollars. A sustaining member who contributes annually at least five hundred dollars shall be designated as a Patron of the Society.

Section 7. Persons elected to ordinary membership as nominees of sustaining members or institutional contributing members under the provisions of Article VI, Sections 3–4, shall not be required to pay an initiation fee. They shall not be required to pay dues, so long as it is agreed that they are designated by the sustaining member or institutional contributing member under these provisions. If a nominee of a sustaining

member or an institutional contributing member later becomes a dues-paying member, he shall pay dues at the rate of six dollars a year for the remainder (if any) of his first three years of membership, and eight dollars a year thereafter.

Section 8. If the annual dues of any member remain unpaid beyond a reasonable time, the Board of Trustees shall remove his name from the list of members, after due notice.

Section 9. Any member not in arrears of dues may become a life member on the payment of a sum to be determined in accordance with actuarial principles. A life member shall have for life the status and privileges of an ordinary member, without further payment of dues.

Section 10. After retirement from active service on account of age, any member who is not in arrears of dues and with membership extending over at least twenty years may—by giving proper notification to the Secretary—have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings, but not the *Bulletin*.

Article VIII. Meetings

Section 1. The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following. Notice of the time and place of this meeting shall be mailed by the Secretary or an Associate Secretary to the last known post office address of each member of the Society. The times and places of the Annual and other meetings of the Society shall be designated by the Council. No matter of general business shall be considered at any meeting of the Society except the Annual Meeting, without the recommendation of the Council.

Section 2. The President may call a meeting of the Council whenever the affairs of the Society make it desirable; he shall call a meeting of the Council on written request of five of its members.

Section 3. The Board of Trustees shall have its regular Annual Meeting on the first day of the Annual Meeting of the Society. Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days' notice of such meeting mailed to the last known post office address of each trustee. He shall call a meeting upon the receipt of a written request of two of the trustees. Meetings for the transaction of business may also be held by common consent of all the trustees.

Section 4. Papers intended for presentation at any meeting of the Society shall be passed upon in advance by a program committee appointed by or under the authority of the Council; and only such papers shall be presented as shall have been approved by such committee. Papers in form unsuitable for publication, if accepted for presentation, shall be referred to on the program as preliminary communications or reports.

Article IX. Publications

Section 1. The Society shall publish an official organ called the *Bulletin of the American Mathematical Society*. It shall publish a journal called the *Transactions of the American Mathematical Society*, the object of which shall be to make known important researches presented at meetings of the Society. It shall publish a periodical called *Mathematical Reviews* containing abstracts or reviews of current mathematical literature. It shall publish a series of volumes called *Colloquium Publications* which shall embody in book form new mathematical developments. It shall also cooperate in the conduct of the *American Journal of Mathematics*.

Section 2. The editorial management of the *Bulletin*, *Transactions*, *Mathematical Reviews*, and *Colloquium Publications*, and the participation of the Society in the

management of the *American Journal of Mathematics* shall be in charge of the respective Editorial Committees as provided in Article III, Section 1.

ARTICLE X. AMENDMENTS

These By-Laws may be amended or suspended at any meeting of the Society on recommendation of the Council and by a two-thirds vote of the members present, provided notice of such proposed action and of its general nature shall have been given in the call for such meeting.

Following the business meeting Professor H. F. Bohnenblust gave an address entitled *Partially ordered linear spaces*. During the latter part of the afternoon there were two sessions, one for Geometry, Professor Virgil Snyder presiding, and one for Algebra and Analysis at which Professor E. L. Post presided.

Titles and cross references to the abstracts of papers read at this meeting follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 9 were read before the section for Analysis; those numbered 10 to 18 before the section for Algebra, Topology, and Statistics; those numbered 19 to 22 in the section for Geometry; those numbered 23 to 27 in the section for Algebra and Analysis; papers numbered 28 to 54 were read by title. Paper 21 was read by Professor Kasner. Dr. Bers was introduced by Dr. Abe Gelbart, Dr. Weinstein by Dr. H. E. Robbins, Dr. Snapper by Professor J. H. M. Wedderburn, Mr. Rosenbloom by Professor J. R. Kline, Dr. Bergman by Dr. D. C. Spencer, Dr. Kober by Professor Einar Hille, and Dr. Wundheiler by Professor D. J. Struik.

1. R. P. Boas: *A note on functions of exponential type.* (Abstract 47-3-129.)

2. P. R. Halmos: *On the decomposition of measures.* (Abstract 47-3-133.)

3. Abe Gelbart: *On the behavior of a function of two complex variables in the neighborhood of an isolated essential singularity.* (Abstract 47-3-131.)

4. I. J. Schoenberg: *On absolutely convex functions.* (Abstract 47-5-245.)

5. George Polya: *On converse gap theorems.* (Abstract 47-3-139.)

6. Dorothy Maharam: *On measure in abstract sets.* (Abstract 47-3-136.)

7. Lipman Bers: *Analytic functions of two complex variables in domains bounded by two analytic hypersurfaces.* (Abstract 47-3-128.)

8. H. N. Laden: *On Kryloff-Stayermann interpolation.* (Abstract 47-3-135.)

9. Alexander Weinstein: *On the decomposition of a Hilbert space by its harmonic subspace.* (Abstract 47-5-252.)

10. Ernst Snapper: *Structure of linear sets.* (Abstract 47-3-124.)

11. P. C. Rosenbloom: *Post algebras: I. Postulates and general properties.* (Abstract 47-3-123.)

12. Hassler Whitney: *Geometric methods in combinatorial topology.* Preliminary report. (Abstract 47-3-180.)

13. Oystein Ore: *Transformation of sets.* (Abstract 47-3-122.)

14. Shizuo Kakutani: *A generalization of Brouwer's fixed-point theorem.* (Abstract 47-3-177.)

15. Archie Blake: *The exploratory determination of statistical distributions.* (Abstract 47-3-169.)

16. Abraham Wald: *On testing statistical hypotheses concerning several unknown parameters.* (Abstract 47-5-275.)

17. W. G. Madow: *The distribution of the general quadratic form in normally distributed random variables.* (Abstract 47-3-170.)

18. Jacob Wolfowitz: *Tests of statistical hypotheses where the distribution forms are unknown.* (Abstract 47-3-172.)

19. Aaron Fialkow: *The foundations of the conformal differential geometry of a subspace.* (Abstract 47-3-156.)

20. Peter Scherk: *On real closed curves of order $n+1$ in projective n-space.* II. Preliminary report. (Abstract 47-3-165.)

21. Edward Kasner and J. J. DeCicco: *Lineal element transformations which preserve isothermal families.* (Abstract 47-3-161.)

22. J. M. Feld: *The geometry of whirls and whirl-motions in space.* (Abstract 47-3-155.)

23. Howard Levi: *On the ideal theory and structure of differential polynomials.* (Abstract 47-3-120.)

24. Marshall Hall: *A problem in partitions.* (Abstract 47-3-118.)

25. Irving Kaplansky: *Maximal fields with valuations.* (Abstract 47-1-5.)

26. Stefan Bergman: *Numerical methods for conformal mapping of polygonal domains.* (Abstract 47-3-148.)

27. E. R. Kolchin: *On the basis theorem for differential systems.* Preliminary report. (Abstract 47-3-119.)

28. E. F. Beckenbach: *On almost subharmonic functions.* (Abstract 47-3-126-*t*.)

29. Stefan Bergman: *On a class of linear operators applicable to functions of a complex variable.* (Abstract 47-3-127-*t*.)

30. L. M. Blumenthal and G. E. Wahlin: *On the spherical surface of smallest radius enclosing a bounded subset of n-dimensional euclidean space.* (Abstract 47-3-150-*t*.)

31. A. H. Clifford and Saunders MacLane: *Factor sets of a group in its abstract unit group.* (Abstract 47-3-117-*t*.)

32. A. H. Clifford: *Semigroups admitting relative inverses.* (Abstract 47-3-116-*t*.)

33. J. J. DeCicco: *Isodeviate systems of geodesic series.* (Abstract 47-3-152-*t*.)

34. M. C. Foster: *Note on autopolar surfaces.* (Abstract 47-3-157-*t*.)

35. Paul Erdös and Joseph Lehner: *On the distribution of the number of summands in the partitions of a positive integer.* (Abstract 47-3-173-*t*.)

36. H. L. Garabedian: *Hausdorff methods of summation which include all of the Cesàro methods.* (Abstract 47-3-130-*t*.)

37. H. H. Goldstine: *The prolongment and representation of multilinear functionals.* (Abstract 47-3-132-*t*.)

38. F. B. Jones: *Monotonic collections of peripherally-separable connected domains.* (Abstract 47-3-176-*t*.)

39. Edward Kasner and J. J. DeCicco: *Conformal geometry of differential equations of first order.* (Abstract 47-3-158-*t*.)

40. Edward Kasner and J. J. DeCicco: *Conformal geometry of velocity systems.* (Abstract 47-3-159-*t*.)

41. H. Kober: *On a theorem of Schur and on fractional integrals of purely imaginary order.* (Abstract 47-5-228-*t*.)

42. Don Mittleman: *Theory of ortho-family: A generalization of natural family.* (Abstract 47-3-163-*t*.)

43. Oystein Ore: *Theory of monomial groups.* (Abstract 47-3-121-*t*.)

44. H. A. Rademacher: *Ramanujan's identities under modular transformations.* (Abstract 47-3-175-*t*.)

45. Raphaël Salem: *On absolute convergence of trigonometrical series.* I. (Abstract 47-3-141-*t*.)

46. Raphaël Salem: *On absolute convergence of trigonometrical series.* II. (Abstract 47-3-142-*t*.)

47. Henry Scheffé: *An inverse problem in correlation theory.* (Abstract 47-3-171-*t*.)

48. J. A. Shohat: *On Bernoulli numbers and polynomials.* (Abstract 47-3-145-*t*.)

49. M. F. Smiley: *Betweenness in general lattices.* Preliminary report. (Abstract 47-5-201-*t*.)

50. I. J. Schoenberg: *On integral representations of completely monotone and related functions.* (Abstract 47-3-143-*t*.)

51. R. C. Stephens: *On abstract mappings.* (Abstract 47-3-179-*t*.)

52. Alvin Sugar: *Postulates for the calculus of binary relations in terms of a single operation.* (Abstract 47-3-168-*t*.)

53. G. T. Whyburn: *On mappings into the circle.* (Abstract 47-3-181-*t*.)

54. Alexander Wundheiler: *Abstract algebraic definition of an affine vector space.* Preliminary report. (Abstract 47-3-166-*t*.)

T. R. HOLLCROFT,
Associate Secretary

WILLIAM CASPAR GRAUSTEIN—IN MEMORIAM

William Caspar Graustein was born in Cambridge, Mass., November 15, 1888. He graduated from Harvard, Magna cum Laude in 1910. He spent the next year in the same place doing graduate work in mathematics, his main interest being then, as ever afterwards, in geometry. He received his first initiation into the higher branches of that subject by reading Klein's incomparable 1893 lectures on "Höhere Geometrie." As a matter of fact this nearly had a most disastrous effect, for when Study in Bonn heard that his prospective pupil had been reading Klein, he wrote, "So ist er ganz und gar verdorben." However, Study relented sufficiently to receive him in the autumn of 1911, thereby initiating a close personal and intellectual intimacy that lasted many years, and had the deepest influence on the whole of Graustein's subsequent career.

He took the doctor's degree, Summa cum Laude in July, 1913, and returned at once to Harvard as instructor in mathematics. He was then promoted to an assistant professorship in the newly formed Rice Institute in Houston, Texas. He joined his Cambridge friend Griffith Conrad Evans in building up there, with the warm approval of the President Edgar Odell Lovett, the most advanced mathematical centre south of Mason and Dixon's line. This lasted for four fruitful years, then came the war. Graustein was bound he would serve his country, in spite of official discouragement on account of his name and antecedents, and serve he did in the Ordnance Department at the proving ground at Aberdeen, Maryland, eventually becoming a first lieutenant.

He returned to Harvard in the autumn of 1919 to help in the congenial task of rehabilitating the Division of Mathematics, which in one year had lost three of its most valuable members through death and transfer. He worked mightily at that task till the day of his death. He was chairman of the Division from 1932 to 1937, and Assistant Dean of the Faculty of Arts and Sciences 1939–41. But he served mathematics outside his own university as well, acting for twelve years (1924–36) as an Associate Editor, and for five years (1936–41) as one of the Editors, of the Transactions of the American Mathematical Society, and chairman of the very busy organizing committee of the proposed 1940 International Mathematical Congress in 1938. He was killed in a motor accident, January 22, 1941.

Graustein's doctoral thesis, written with Study, was entitled "Eine reelle Abbildung analytischer komplexer Raumkurven." For a number of years Study had been interested in the geometry of the complex

domain. In three-space we may have varieties depending analytically on any number from one to five of real parameters. But it is always heartening to have something real to represent complex elements, and many writers had occupied themselves with this problem of representation. Study had recently flirted with one of the methods, which was based on an idea initiated by Poncelet, but more closely associated with the writing of a vituperative Frenchman, F. Maximilien Marie. It consists in replacing each complex point by an ordered pair of real points, the closest pair of the elliptic involution whose double points are the given point and its conjugate. This conjugate is represented by the same pair in opposite order. Analytically the point with rectangular Cartesian coordinates x^i is represented by the ordered pair

$$y^i = \frac{x^i + \bar{x}^i}{2} + i\,\frac{x^i - \bar{x}^i}{2}; \qquad \omega^i = \frac{x^i + \bar{x}^i}{2} - i\,\frac{x^i - \bar{x}^i}{2}.$$

When it comes to representing simple loci like lines, circles and planes, there are a distressing number of special cases which must be handled separately; the isotropic elements cause trouble. The points of a complex space curve depend on two real parameters, and will be represented by the points of two real surfaces connected by an analytic relation. Graustein shows that:

A. They are translation surfaces.

B. There is a first and a second system of curves on the one so related to a first and a second system on the other that at corresponding points the tangent to the first curve on either surface is parallel to that of the second curve on the other.

C. The relation is area-preserving, but not conformal.

Graustein retained his interest in complex geometry for a number of years. In Volume 16 of the Transactions he takes up the question of when pairs of complex elements in euclidean three-space can be carried into one another by real motions. The criterion depends on certain real invariants, as the squared distance from a complex point to its conjugate. In the twenty-fourth volume of the same journal he generalizes the Marie representation. What is the most general representation where a complex point (z) is represented by a real ordered pair $(x)(y)$ it being required that the conjugate point $(\bar{z})$ shall be represented by $(y)(x)$, a real point shall be represented by itself counted twice, the functions involved shall be analytic, and the relations involved shall be invariant under some specified group. Two years later in Volume 26 of the Annals of Mathematics he relates the

generalized method of representation to the main problem of his thesis.

A complex curve is represented by two real surfaces in one-to-one correspondence with correspondence of parallelism. This idea led Graustein to make a further study of the relation of two surfaces when the normals at corresponding points are parallel, and in Volume 23 of the Transactions he published a paper on "Parallel Maps of Surfaces." If we have a one-to-one correspondence between two surfaces, neither of which is developable, of such a sort that normals at corresponding points are parallel, they have the same spherical representation. If a point P' corresponds to a point P, there is a projective relation among the tangents at P' connecting each with that which is parallel to the tangent at P corresponding to the given one. When the relation is non-parabolic, that is to say, has two self-corresponding elements, the projective invariant I of this relation is fundamental in the transformation. Graustein shows that it is equal to the cross ratio which the pair PP' make with the corresponding focal points in the congruence of lines connecting corresponding points. This beautiful result leads to many interesting theorems. The author wrote five other short papers dealing with mapping and applicability.

The development of the tensor calculus of Ricci and Levi-Civita introduced a modification in differential geometry as profound as that produced nearly a century before by the publication of Gauss' "Disquisitiones Generales." Many geometers threw themselves entirely into the new work; Graustein was more cautious, he recognized the advantages in the new notations, new points of view and new techniques, especially when more than three dimensions were involved. But what attracted him most was the invariant or covariant character of the new processes, and that led him to the idea of developing methods on the more classical lines, which seemed to him to go really deeper into the geometrical questions involved. The first result of this study was his "Méthodes Invariantes dans la Géométrie Infinitésimale," crowned by the Belgian Academy of Sciences in 1929. A year later appeared in this Bulletin, Volume 36, an address delivered by request before this Society, "Invariant Methods in Classical Differential Geometry."

What sort of things are invariant under the transformations of differential geometry? First of all Jacobians are. A vector normal to a surface is not concerned with the choice of parametric curves on the surface, and the three components of such a vector are proportional to the three Jacobians $\partial(x^i x^j)/\partial(uv)$. Again we have three funda-

mental quadratic differential forms, that which gives the directions of the isotropic lines, that which gives the asymptotic lines, and that which gives the isotropic directions on the spherical representation. Their algebraic invariants will give geometric invariants of the surface. Beyond these there is an invariant process called directional differentiation. This was, I think, first developed by Lamé. Graustein uses it with telling effect. The directional derivative of a point function f in the direction of a given element of arc is

$$\lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s} = \frac{\partial f}{\partial s}.$$

The great trouble is that the order of differentiation is material when we perform the process twice

$$\frac{\partial^2 f}{\partial s \partial s'} \neq \frac{\partial^2 f}{\partial s' \partial s}.$$

He turns the difficulty by introducing a new operator ∇. If s and s' are arcs on an orthogonal system of curves,

$$\frac{\nabla f}{\nabla s} = \frac{\partial f}{\partial s} + \frac{1}{\sigma'} f, \qquad \frac{\Delta f}{\Delta s'} = \frac{\partial f}{\partial s'} - \frac{1}{\sigma} f$$

where $1/\sigma$ is the geodesic curvature of s and $1/\sigma'$ that of s'. It then appears that

$$\frac{\nabla}{\nabla s} \frac{\partial f}{\partial s'} = \frac{\nabla}{\nabla s'} \frac{\partial f}{\partial s}.$$

The Gauss and Mainardi-Codazzi equations are easily found in terms of these new symbols.

It is clear that such invariant methods, though first developed for the case of euclidean measure in three-space, can be widely generalized. Graustein did just this in the thirty-sixth volume of the Transactions. Here we have a forty-page article on the geometry of Riemann spaces. The basis of directional differentiation is a mixed tensor given by the equation

$$\frac{\partial}{\partial s^q} \frac{\partial f}{\partial s^p} - \frac{\partial}{\partial s^p} \frac{\partial f}{\partial s^q} = B_{ijk} \frac{\partial f}{\partial s^k}.$$

Another development of these directional methods appeared two years earlier in the same journal, in an article entitled "Parallelism and Equidistance in Differential Geometry." Consider two curves C and $C'+\Delta C'$ on a surface. What shall we mean by saying that they

are equidistant with regard to a one-parameter family of curves C'? Clearly that the arcs of the latter bounded by the first two are all equal. But when they are not equal we should like to develop some measure of their spreading. If the curves C are given by the values of the function f Graustein defines as the "distantial spread" of the curves C with regard to the curves C'

$$\frac{1}{b'} = -\frac{\partial}{\partial s}\log\left|\frac{\partial f}{\partial s'}\right|.$$

I mention in passing a personal regret that the author found it necessary to mar such a pretty conception with such a cacophonous name. Analogous to this is the angular spread, defined by the equation

$$\frac{1}{a} = \frac{1}{\rho} + \frac{d\omega}{ds}$$

where ω is the properly measured angle between that curve C on which the distances are measured and the various curves C' while $1/\rho$ is the geodesic curvature. The two types of spreads are closely connected, and have many pretty properties. $1/\rho$ vanishes in the case of Levi-Civita parallelism. The two are connected by interesting relations.

I will give it as my personal opinion that Graustein's work dealing with invariant methods was his most important contribution to mathematical science. His last published paper, in the forty-seventh volume of the Transactions, was somewhat different. It gave a complete and beautiful solution of the difficult problem of harmonic minimal surfaces. This had been only incompletely handled by previous writers.

Any notice of Graustein's life work should include some mention of his textbooks. The first of these, "Plane and Solid Analytic Geometry" written in collaboration with W. F. Osgood appeared in 1921. It is a thoroughly usable introductory text, with adequate exercises. It has never seemed to me, however, that the book was entirely at peace with itself. When a distinguished analyst and an enthusiastic geometer work together, if both are good teachers, the result may well be a good book, but it may also lack the inner consistency which it would have possessed had either author written it alone. No such charge can be brought against Graustein's "Introduction to Higher Geometry" which first saw the light in 1930 after many years of most careful preparation. Soon after W. E. Byerley returned to Harvard in 1876 he introduced a course in "Modern

Methods in Analytic Geometry" drawing its inspiration from the lectures by Evan W. Evans that he had heard while he was at Cornell. That course became at once one of the fixtures in the Harvard mathematical curriculum, successive teachers modified it to suit their own ideas and Graustein gave it repeatedly. This excellent book represented the results of his experience. The first three quarters are devoted to a careful presentation of the projective geometry of points, lines, and conic sections, combining geometric and algebraic methods easily and naturally. Then fifty pages are given to the circle, and the geometry of inversion. This is prettily done. I cannot help feeling, however, that it should have been postponed to the end of the book, as the author seems to have introduced it because he felt, and rightly, that no well trained geometer should be ignorant of these things, rather than because it fitted naturally with the rest of the work. The last chapter of forty pages, deals all too briefly with the projective geometry of three-dimensional space.

Graustein's last textbook "Differential Geometry" appeared in 1935. This also represents the results of many years of teaching an introductory course in the subject. It is classical and conservative in spirit, great use is made of vector methods, and Study's vector notation. Near the end we have a twenty page introduction to the absolute differential geometry. It is interesting to contrast this book with the recent one on the same subject by Eisenhart, published just five years after. One has the impression that Graustein introduced some ideas of absolute geometry as a matter of conscience, he did not like the idea that a student who had finished the book should have no knowledge whatever of the more modern technique. Eisenhart introduces tensor notation as soon as he can and feels himself cramped as long as he is confined to three dimensions.

It is not the function of this Bulletin to carry long reviews of textbooks, but it is eminently proper to bring to the memory of the members of our Society beautifully written works of a great teacher and great friend of all who teach or learn mathematics.

WILLIAM C. GRAUSTEIN

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3. Review: *Dickson's Linear Algebras*, Bulletin, July 1915, pp. 511–522.
4. *On the geodesics and geodesic circles on a developable surface*, Annals, March 1917, pp. 132–138.
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6. *Note on isogenous complex functions*, Bulletin, July 1918, pp. 473–477.
7. *Parallel maps of surfaces*, Transactions, April 1922, pp. 298–332.
8. *Representations of a complex point*, Transactions, December 1922, pp. 245–254.

9. *Spherical representation of conjugate systems*, Annals, December 1922, pp. 89–98.

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11. *Determination of a surface*, Journal, October 1923, pp. 294–314.

12. *Applicability with preservation of both curvatures*, Bulletin, January 1924, pp. 19–23.

13. *Isometric W-surfaces*, Transactions, April 1924, pp. 176–204.

14. *A necessary and sufficient condition that two surfaces be applicable* (with B. O. Koopman), Transactions, July 1924, pp. 369–372.

15. *The scientific work of Joseph Lipka*, Bulletin, July 1924, pp. 352–356.

16. *Real representations of analytic complex curves*, Annals, September 1924, pp. 131–143.

17. Review: *Eisenhart's Transformation of Surfaces*, Bulletin, October 1924, pp. 454–460.

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20. *An invariant of a general transformation of surfaces*, Bulletin, July 1926, pp. 357–364.

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22. *A geometrical problem in which order of choice is important*, Monthly, October 1928, pp. 412–415.

23. Review: *Weatherburn's Differential Geometry*, vol. 1, Bulletin, November 1928, pp. 785–786.

24. Belgian memoir (1929): *Méthodes Invariantes dans la Géométrie Infinitésimale des Surfaces*, pp. 1–96.

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26. *The linear element of a Riemannian V_n*, Journal, April 1930, pp. 351–356.

27. *Invariant methods in classical differential geometry*, Bulletin, August 1930, pp. 489–521.

28. *On the average number of sides of polygons of a net*, Annals, January 1931, pp. 149–153.

29. Review: *Weatherburn's Differential Geometry*, vol. 2, Bulletin, November 1931, pp. 796–797.

30. *Parallelism and equidistance in classical differential geometry*, Transactions, July 1932, pp. 557–593.

31. *The geometry of Riemannian spaces*, Transactions, July 1934, pp. 542–585.

32. *Applicability with preservation of both curvatures*, Duke Journal, June 1936, pp. 177–191.

33. *A new form of the four-vertex theorem*, Monatshefte für Mathematik und Physik, vol. 43, Vienna, 1936, pp. 381–384.

34. *Extensions of the four-vertex theorem*, Transactions, January 1937, pp. 9–23.

35. Review: *Petermann and Hagge's Gewachsene Raumlehre*, Monthly, June–July 1937.

36. *The four-vertex theorem of a certain type of space curves* (with S. B. Jackson), Bulletin, October 1937, pp. 737–741.

37. *Harmonic minimal surfaces*, Transactions, March 1940, pp. 173–206.

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"Introduction to Higher Geometry," 1930.

"Differential Geometry," 1935.

J. L. COOLIDGE

BOOK REVIEWS

Bibliography on Orthogonal Polynomials. By J. A. Shohat, Einar Hille, and J. L. Walsh. Washington, D. C., National Academy of Sciences, 1940. 9+203 pp. $3.00.

This bibliography is concerned principally with Hermite, Legendre, Laguerre, Jacobi, Tchebycheff, and related polynomials. These polynomials, which enter into many phases of pure and applied mathematics, happen to be both important and interesting. Since the time of Legendre (1752–1833) new applications have been requiring new advances in the theory of the polynomials, and new advances in the theory have been finding new applications. Hence it is inevitable that the literature of the subject has grown to vast proportions and will continue to grow. One who, by reason of preference or necessity, wishes to become acquainted with the whole theory or with some part of it is embarrassed not only by the extensiveness of the literature but also by the fact that so many different persons have made contributions in so many different books and periodicals. It is fortunate that the National Research Council of the National Academy of Sciences has collaborated with the three authors to bring forth this bibliography.

The book begins with an alphabetical list of complete titles of 332 periodicals. These periodicals are thereafter referred to by number. This necessitates repeated reference to the list of titles, but it eliminates confusion which results from abbreviations.

The second part of the book gives a code index. Some Roman capitals represent special classical orthogonal polynomials (OP); for example H represents Hermite OP. Other Roman capitals specify the domain of orthogonality and the character of the weight-function. Lower case Greek letters refer to topics, properties, applications, etc.; subheadings are lower case Roman letters and then Arabic numbers. For example α refers to general properties of OP; αd to zeros of OP; Hαd to zeros of Hermite OP; and H$\alpha d4$ to bounds for the zeros of Hermite OP. The symbol β refers to expansions of functions; $\beta a1, \beta a2, \cdots, \beta a19$ refer to 19 different classes of functions (L, L^2, L^p, continuous, bounded variation, etc.) to be expanded; and $\beta b1, \cdots, \beta b19$ refer to properties of expansions. For example $\beta b14.1, \cdots, \beta b14.9$ refer to 9 different types of convergence of expansions such as uniform, absolute, almost everywhere, and in mean; and $\beta b15.1, \cdots, \beta b15.6$ refer to summability of expansions. The symbol γ refers to general series of OP not necessarily resulting from expan-

sion of functions; δ to mechanical quadratures; ϵ to interpolation; ζ to approximations; η to the moment problem; θ to probability and statistics; and the code continues until ω is reached.

The third and main part of the book (pp. 25–184) lists 1952 articles by about 600 authors. The full title of each article is given exactly as it appears with the article except in the case of non-Latinic alphabets, when a translation is given. Following each title is an abstract in code. For example, the fourth entry under E. Hille is:

A class of reciprocal functions. [36] (2) 27 (1926) 427–64.

H:αb4−b7−b10−b13−c−d1−d3−d4−d7−f−f3−g−i−j−l1 m1.1−m1.2−n2−βa1−a5−b12−b14.1−b14.2−b14.3−b14.4 −b14.8−b14.9−b15.1, θa2, κb, μ, π, ρ.

It takes only a short time to find that the [36] refers to the Annals of Mathematics and to decode the abstract to obtain a clear and precise statement of the topics discussed in the paper.

The fourth part of the book lists 89 books and 39 theses containing theory of OP. Some of these are abstracted and some are not.

The fifth and last part of the book is a topical index which lists works on (i) Hermite OP; (ii) Jacobi OP; (iii) Laguerre OP; (iv) Legendre OP; (v) Applications to theory of approximation, probability and statistics, and mathematical physics; (vi) Series in OP; (vii) Interpolation, mechanical quadratures; (viii) Moment problem; (ix) OP in the complex domain; and (x) Tables and graphs. One feels that these lists are long; there are for example more than 500 papers on series in OP of which the three authors of this bibliography are collectively responsible for 29. Checking over 500 abstracts to find, for example, papers on uniform convergence of series in Hermite polynomials sounds like a formidable task; but actually this task is very much simplified by the code form of the abstracts. One simply looks for Hβb14.1 without being required to read hundreds of pages of verbose abstracts.

So far as the reviewer can determine, the bibliography is accurate and is complete in the sense that it contains all titles which it certainly should contain. There are naturally many books and papers with application to OP which are not cited; for example some on general theory of orthogonal functions which apply to polynomials as well as to other orthogonal functions; some on trigonometric and power series; and some on Sturm-Liouville theory. But to start listing all these and other works which bear upon some ramification of the theory of OP would lead to bibliography without end, and one must

feel that the authors have shown good judgment in delimiting their bibliography.

The standard of excellence and usefulness of this book by three capable authors is such that the book will be a valuable adjunct to libraries and to the research equipment of those who work with *OP*. It is appropriate that the book should be used as a model for future bibliographies in other subjects.

RALPH P. AGNEW

Theoretical Hydrodynamics. By L. M. Milne-Thomson. London, Macmillan, 1938. 552 pp. $11.25.

A very comprehensive mathematical treatment of the theory of fluid motion is contained in this text, which presents the lectures by the well known author on this subject to the junior members of the Royal Corps of Naval Constructors at Greenwich. The material presented consists mainly of the theory of the perfect fluid motion, for the first time based consistently on vector notation throughout the text, which thus becomes very concise and brief in the details of mathematical deduction. The great advantages of this form of mathematical writing for this field are obvious to anyone who surveys the wealth of material given in this text. In justice to the author not all the saving in space should be attributed to the use of vector notation; however, instead a considerable amount can be ascribed to the experienced and able use of the descriptive text which is brief but complete in all details. However, the physical interpretations given seem to be treated somewhat too briefly for engineers, and experimental and practical applications are outside of the scope of this theoretical treatment.

A brief review of the contents will bring out the structure of the book. Chapter I contains elementary problems of great variety based on Daniel Bernoulli's theorem. The mathematical tools are introduced in Chapter II on vector analysis, which is followed by the discussion of general properties of fluid motion and such phases of two-dimensional flow as can be treated without recourse to the complex variable. Chapter V introduces the latter and opens up the main part of the book, Chapters VI–XIV, dealing comprehensively with two-dimensional motion in all its aspects from the standpoint of the complex variable and conformal mapping. In addition to the chapter headings of streaming motion, aerofoils, sources and sinks, moving cylinders, theorem of Schwarz and Christoffel, the wake, rectilinear vortices, we find also jets and currents and waves treated extensively in this

part. The succeeding four chapters are taken up with three-dimensional phenomena and related mathematical problems under the headings of Stokes' stream function, spheres and ellipsoids, solid moving through a liquid, vortex motion. The last chapter is devoted to the introduction of the equations of motion for viscous fluids and concludes with a brief description of boundary layer theory.

While this book can be recommended as a text in classical hydrodynamics for advanced graduate student engineers, it will take a more important place as a reference work and as a source of many original approaches in the manner of presentation for those teaching the subject. The over five hundred exercises ranging from easy to very difficult should prove very interesting since many were taken from the official examinations for Constructor Lieutenants at the Naval College and for the degree of M. Sc. at the University of London.

A. T. IPPEN

Convergence and Uniformity in Topology. By J. W. Tukey. Annals of Mathematics Studies, no. 2. Princeton, University Press, 1940. 9+90 pp. $1.50.

The extension of metric methods to non-metrizable topological spaces has been a principal development in topology of the past few years. This has occurred in two directions: one through a rebirth of interest in Moore-Smith convergence due to results of Garrett Birkhoff, and the other through the concept of uniform structure due to André Weil. In this pamphlet these ideas and their interrelations are given a full and detailed treatment. Many of the results are new. This is likewise true of the point of view and much of the mechanism.

Chapters I and II are concerned with set theory, partially ordered sets, Zorn's lemma, and directed sets. In particular a *stack* (=the set of finite subsets of a given set, ordered by inclusion) is a directed set. Directed sets are classified into cofinally equivalent types, and these are found to be partially ordered. Chapter III introduces the *phalanx* (a function from a stack to a topological space T). It is proved that the topology of T is describable by the convergence of its phalanxes. Chapter IV considers compactness and equivalent properties in terms of phalanxes. The *biggest* compactification of a space is defined and constructed from ultraphalanxes. Chapter V is concerned with coverings of a space (by open sets), and the equivalence of the existence of families of coverings to the existence of metrics and pseudo-metrics. This leads naturally to the notion of *uniform structure* (Chapter VI). A uniform structure is a family $\{U\}$

of coverings of T such that, for any two, U_1, U_2, there is a third, U_3, such that, for each $u_0 \in U_3$, the sum of the u's in U_3 meeting u_0 lie in a set of U_1 and in a set of U_2. Such a structure exists if and only if T is completely regular. This notion generalizes the notion of metric. A given space may have various inequivalent uniformities; a compact space has only one. The concept of completeness is extended, and it is proved that any space with a uniform structure may be completed. Chapter VII carries over these notions from a collection of spaces to their product. An extension of Tychonoff's embedding theorem is proved. Chapter VIII presents a number of examples.

An extensive symbolism makes the reading difficult—especially so when meeting one of the numerous misprints or omissions.

N. E. STEENROD

Formal Logic. By Albert A. Bennett and Charles A. Baylis. New York, Prentice-Hall, 1939. 17+407 pp.

It is well known that logic, which for a long time remained in practically the same form in which Aristotle left it, has lately been undergoing a revolution. The cause of this phenomenon is, essentially, the impact of mathematics upon logic. It has been realized for some time that, by using mathematical methods, the statement of logical principles can be greatly simplified; likewise their scope can be extended so as to take in types of reasoning—such as those dealing with relations—which were only awkwardly handled by the traditional methods. The result has been the creation of an entirely new formal framework of great power and elegance.

As in many such revolutions a long time has had to elapse between the discovery of the new ideas and their appearance in elementary textbooks. Although the revolution in logic has been going on for nearly a century, yet it is only within about the last decade that textbooks of general logic have shown appreciably the effects of the new ideas. But since 1930 there have appeared a number of books of that character in which this effect has been marked. These include books by Stebbing, Cohen and Nagel, Eaton, Chapman and Henle, and a recent book by Churchman. None of these books contains a complete revamping of the subject, but every one is something of a compromise between the old and the new; nevertheless the amount of information concerning the new doctrine is considerable.

The present book belongs to the category just described. It aims "to provide a simple and clear survey of the field of formal logic synthesizing classical and modern developments into a unified treat-

ment." It is the joint work of a mathematician and a philosopher, both of whom identified themselves with the new movement to the extent of becoming officers in the Association for Symbolic Logic. Under these circumstances a deeper penetration by the new ideas is to be expected.

The book begins with a chapter on the nature of logic. This is extremely well done; there is nothing comparable to it in any of the other books at the reviewer's disposal. A feature of it is the inclusion of examples of relatively complicated reasoning processes to be carried out informally. This is followed by a chapter on logical symbolism, in which the nature of symbols, and also of propositions, is discussed. After this follow two chapters containing a philosophical discussion of individuals, concepts, relations, classes, etc. The next two chapters are devoted to the traditional analysis of propositions and syllogisms. Boolean algebra, with a class interpretation, is then taken up in a chapter entitled "Formal Non-syllogistic Class Inference." After a chapter on classification, definition, and descriptions there follow three chapters, with a distinctly modern flavor, on the propositional algebra, the theory of propositional functions, and the formulation of logical deductive systems. The book closes with two chapters on probability and scientific method, occupying together about fifty pages.

The above outline will give an idea of the scope and general plan of the book. Let us now consider a few criticisms.

The reviewer is somewhat disappointed in regard to the treatment of the traditional logic. The procedures of the Boolean class algebra are simpler than those of the traditional logic and include them as special cases. Consequently, with the introduction of the former logic the latter has become practically useless. Yet in spite of this the authors devote a considerable portion of their book to developing the traditional logic along essentially traditional lines; in particular they establish the valid moods by giving a table showing the possible conclusions in each of 64 possible cases, and then saying that the reader may check the table by means of the Veine diagrams. There should be a Society for the Prevention of Cruelty to Students to stop this sort of thing!

The foregoing must not be supposed to mean that traditional logic has no uses whatever. There are in fact two main uses, neither of them practical. In the first place a book like this will be studied largely by philosophers; and in philosophy it seems that the history of the subject is especially important. From this point of view Aristotelian logic will be interesting to philosophers for a long time to come. Again,

the derivation of traditional logic on the basis of an underlying Boolean algebra is a useful illustration of the technique of the latter. The first of these purposes will be equally well served, and the second vastly better, if traditional logic is treated after Boolean algebra, rather than before. Then the syllogistic rules can be derived, e.g. in ways suggested by the reviewer in *Mind*, vol. 45 (1936), pp. 209–216.

In connection with the above criticism there is another related one. The traditional hypothetical syllogism, dilemma, etc., are treated by the authors in the chapter on syllogisms. It is evident, however, that these topics have to deal with propositions as unanalyzed wholes, and should therefore be treated in an earlier part of the discussion. It would be useful to have all the relations between propositions as such, including those discussed here in Chapter 2, treated together under the heading Propositions. So far Churchman is the only author who has done this (see his Chapter 2).

The discussion should be preceded by a general discussion of significance of formal methods. The present authors postpone this to Chapter 11, at which time the student has become sufficiently sophisticated to tackle the difficulties of applying this idea to the calculus of propositions. But an idea of the nature of formal reasoning, as exemplified in the class calculus, can be presented to the reader at a much earlier stage; just as a quite adequate idea of the nature of logic as such is presented in Chapter 1. Such a discussion is the natural sequel to Chapter 1 as presented; and the reviewer was conscious of a hiatus at that point. A discussion of the sort stated is necessary in order to answer the question of how we can use formal methods to establish logical theorems, such as would be necessary for a development of the theory of the syllogism according to the above lines.

Let us now turn to a criticism of a different kind. In the discussion of concepts and classes the authors introduce at the start two levels of equivalence (and hence of inclusion), viz., strict or logical equivalence, and material or factual equivalence. The distinction is in keeping with a conceptualist-idealist point of view which the authors maintain throughout the work. The reviewer sympathizes with this point of view, and recognizes that for some purposes it may be necessary to consider simultaneously not only two, but even three or more such levels of abstraction. But if such distinctions are introduced into the calculus of classes the result is a mathematical system of great complexity. It seems better, when setting up the formal calculus, to introduce it first with only equality relation; to point out that this equality relation may be interpreted in either way; and to reserve the consideration of a calculus with two or more equality

(or inclusion) relations until a later stage of the inquiry. Generalizations of this kind are not considered in any other elementary text.

To sum up, we have in this book by Bennett and Baylis a textbook on logic, designed for presentation to beginners, and intended as an introduction to modern mathematical, as well as to traditional formal logic. This is a difficult expository problem; and one for which a thoroughly satisfactory solution has not yet been found by anyone. That the book should, under these circumstances, be something of a compromise, is perhaps inevitable. The reviewer has criticized it from an ideal point of view, with reference to the goal to be attained—which, by the way, is of some importance for mathematics; these criticisms are to be taken not as pointing out defects in this book but as suggesting ways in which the next approximation to the goal can be improved. The text is one of great merit; most of the criticisms here made would apply to any similar book the reviewer knows of.

HASKELL B. CURRY

The Theory of Group Characters. By D. E. Littlewood. Oxford University Press, 1940. 8+292 pp.

The theory of group representations and group characters has already been treated very recently in the comprehensive expositions of Hermann Weyl's *The Classical Groups*,[1] and F. D. Murnaghan's *The Theory of Group Representations*. We now have a third treatment. All three differ not only in emphasis but also in the spirit[2] of their approach to the subject.

Littlewood's book is intended by him to give "a simple and self-contained exposition of the theory[3] in relation to both finite and continuous groups, and to develop some of its contacts with other branches of pure mathematics such as invariant theory and the theory of symmetric functions." Thus the first fifty-two pages of his text are devoted to an exposition designed to make it self-contained. Chapter I consists mainly of a discussion of the classical canonical form of a matrix under similarity transformations, the properties of unitary, orthogonal and real orthogonal matrices, the reduction of Hermitian matrices under unitary transformations, and the definition of direct product. Chapter II presents the concept of an algebra and its regular representations, the consequent definition of trace, and the further topics necessary for an understanding of the property

[1] Reviewed in this Bulletin, vol. 46 (1940), pp. 592–595.

[2] Cf. Footnote 4.

[3] Of group *characters*, not of group representations.

that a semi-simple algebra over the complex field may be expressed as a direct sum of total matric algebras. The third chapter includes the topics of permutation groups, the relation of the number of partitions of an integer n to the number of classes in the symmetric permutation group on n letters, and the representations of finite groups by permutation groups. Finally the discussion of the semi-simplicity of a group algebra and the basic theory of group representations are presented in the beginning of Chapter IV.

The somewhat sketchy character of this first part of Littlewood's book would seem to have been dictated by its brevity and to result necessarily in proofs which are neither as complete[4] nor as conclusive as is desirable. Nonetheless, simpler and shorter modern proofs (of the results on matrices and linear algebras which are developed) do exist and could have been employed. They were hardly to be expected, however, in an exposition which uses *spur* for *trace* and *trace* for a special *spur*, which employs *Frobenius algebra* where nearly all others use *group ring* or *group algebra*, which returns to *simple* matric algebra after fourteen years of *total* matric algebra, which presents the theory of group representations so that a major mining operation is necessary to unearth the fundamental *Schur lemma*, and which ignores completely nearly all work of the past fifteen years on matrices and linear algebras.

The remaining portion of the text comprises nearly five-sixths of it, and is devoted to the properties of group characters themselves. They are defined in Chapter IV and a beginning is made there also of the subject matter of Chapter IX, which is a study of the relations between the characters of a group and those of its subgroups. In Chapter V the formula of Frobenius for the characters of the symmetric group is derived and the properties of primitive characteristic units and the Young tableaux are treated. Chapter VI contains a discussion of immanants and S-functions.[5] The author remarks that the S-func-

[4] For example, the demonstration given of the primitive result that a matrix satisfies its characteristic equation uses the similarity theory to prove the result for matrices with distinct characteristic roots and is completed by the statement "if A is any matrix whatsoever we can find a matrix Z such that $A+\mu Z$ has all its characteristic roots distinct and satisfies its characteristic equation. We now take the limit as μ tends to zero, whence A satisfies its characteristic equation."

[5] Richard Brauer has noted the fact, in his review of this book in the Mathematical Reviews, vol. 2 (1941), p. 3, that these functions were obtained by I. Schur in his Berlin thesis of 1941 as the characters of the integral rational representations of the full linear group. See also Brauer's review for certain questions he raises about the validity of Littlewood's discussion of the characters of a group relative to those of its subgroups.

tions may be regarded as generalizations of the coefficient functions in Taylor series and so his Chapter VII consists of a study of relations between S-functions and special series. Chapter VIII presents a method for the numerical computation of the characters of the symmetric permutation group. The main topic of Chapter IX is stated above but we call attention to Littlewood's interpretation of the topic as a study also of the problem of determining the structure of a group with given characters by an analysis of the characters. The last two chapters connect the theory of group characters for finite groups with the theory of continuous groups, and the book closes with an appendix consisting of tables of the characters of the symmetric groups on $m \leqq 10$ letters and of transitive subgroups.

Littlewood's book is thus primarily an exposition in which the major emphasis is on the theory of group characters for its own sake rather than as a tool for other theories. This is contrary to the attitude of the original investigators in the field, and does result in a book which is notable for the immense number of formulae and formidable computations it contains. The author's justification of his attitude lies perhaps in the remark with which he opens his preface: "Since the discovery of group characters by Frobenius at the end of the last century the development of the theory has been so spectacular and the theory has shown such powerful contacts with other branches of mathematics, both pure and applied, that the inadequacy of its treatment by text-books is rather surprising." The reviewer is inclined to regard as exaggeration the words "spectacular," "powerful" and "surprising," and as unjust the word "inadequacy," but hopes with the author that the special properties of group characters which he derives may have some future interesting applications.

A. A. ALBERT

The Theory of Group Representations. By Francis D. Murnaghan. Baltimore, The Johns Hopkins Press, 1938. 369 pp.

Books on group representations come out in waves, it seems. In short succession, three new books on the subject appeared: Murnaghan's Theory of Group Representations, Weyl's Classical Groups, and D. E. Littlewood's Theory of Group Characters. Perhaps, it is worthwhile to compare the general point of view of these new volumes with the books published during the preceding periods in the history of the theory.

First, group representations were treated as a special chapter of the theory of finite groups, furnishing a powerful method for the study

of these groups. The books of Burnside, Blichfeldt, Dickson, Miller, and Speiser belong to this period. Of course, in those days no self-respecting physicist could be seen near the shelves of libraries housing works on group theory. This was the realm of mathematicians, interested in the subject because of its great mathematical importance and its inner beauty.

Suddenly, with the birth of quantum mechanics and modern nuclear physics, everything was changed. There were no self-respecting theoretical physicists anymore who would not know about representations, Schur's lemma, characters, the orthogonality relations, and so forth. The books of this period (Wigner's Gruppentheorie, Weyl's Theory of Groups and Quantum Mechanics, van der Waerden's Gruppentheoretische Methode in der Quantenmechanik) were dedicated half to group theory and half to quantum mechanics, treating the former subject as far as it was needed for the applications to physics. There is no need to describe the immense progress accomplished in this work. Very likely, the books mentioned will be counted among the classics of science.

It is not surprising that the opening of this new field was accompanied by a change of existing values. Certain special groups, the symmetric group and the rotation group suddenly came into prominence. It would not be true to say that they reenacted the story of Cinderella. Such mathematicians as Frobenius, A. Young, I. Schur, Weyl had studied them in detail, but apparently nobody thought of writing books on them, before the applications to physics were found. Of course, the mathematical work was obliterated by the spectacular success of the applications.

What we see today, seems a natural reaction. The mathematicians feel that a theory which admits such applications deserves to be put in the form of books for its own sake. Of course, the needs of physics do not exhaust the mathematical riches. There are more general cases which can be treated, connections to other mathematical theories which can be studied, new questions which can be asked. In this situation, it is hardly an accident that several mathematicians felt the need for a new book on group representations and proceeded to write it, independently of each other. The mathematical theories were treated in all detail, the physical applications were not given. Of course, the three books of Murnaghan, Weyl, and Littlewood differ widely, owing to the different interests of the authors, and their different points of view.

Professor Murnaghan describes the object and scope of his book with the following words: "We have attempted to give a quite ele-

mentary and self-contained account of the theory of group representations with special reference to those groups (particularly the symmetric group and the rotation group) which have turned out to be of fundamental significance for quantum mechanics (especially nuclear physics)."

The book begins with a short discussion of the group concept, of linear spaces, and of matrices. No systematical treatment of these subjects is attempted, while, in the later text, the knowledge of the reader is widened, whenever the need arises. We have a representation of a group G by linear transformations of a fixed "carrier space" S, if with every element A of G a linear transformation M_A of S is associated such that we have $M_A M_B = M_{AB}$ for any two group elements A, B of G. Chapter II deals with the question of reducibility. If all the linear transformations of a representation F leave a subspace S_1 of the carrier space S invariant, then the representation is reducible. It breaks up into two representations F_1 and F_2, where F_1 consists of the linear transformations induced by F in S_1, and F_2 of those induced by F in the projection space of S modulo S_1. We have complete reducibility, when S is the direct sum of two invariant subspaces S_1 and S_2; in this case, F is completely determined by the two lower dimensional representations F_1 and F_2. (It may be remarked that the terminology of the book is not the customary one; what Murnaghan terms "completely reducible" is usually denoted as decomposable while the words "completely reducible" are used in another connection.) Then the fundamental theorem of Burnside is proved, which gives a necessary and sufficient condition for reducibility of representations; after that follows the generalization of this theorem by Frobenius and Schur. In order to handle the question of complete reducibility in important cases, Auerbach's theory of bounded representations is developed. The third chapter is concerned with the representations of finite groups and their characters. Each representation is characterized by its character, i.e., the trace of the linear transformation M_A which represents the general element A of G. The main properties of these group characters are derived. This is about as much as is given of the general theory, the greater part of the book is dedicated to the study of special groups, and it is on these special groups that the emphasis of the book is placed. Two types of finite groups are treated in detail, the symmetric and the alternating permutation groups. An important method for dealing with finite groups is the forming of the average of a function of a general group element. In order to extend this method to compact continuous groups, it is necessary to replace the summation by an integration over the group

manifold. Particular attention has been devoted to this method of group integration for the unitary and the orthogonal groups. The treatment of the representations of the full linear group includes all continuous representations. In the case of the orthogonal group, not only the one-valued representations but also the two-valued spin-representations are thoroughly studied. In all these cases, many interesting properties of the group characters are derived, e.g., the analysis of the direct product of two representations is given. This is one of Professor Murnaghan's own fields of research. Two short chapters on crystallographic groups and on the Lorentz group and semivectors conclude the book.

This description of the contents will show that the "meet" of Murnaghan's book and Weyl's "Classical Groups" is smaller than one might have expected. Actually, on several occasions, friends of the reviewer expressed regret that two so similar books appeared at the same time. It is therefore necessary to emphasize that they are written for quite different classes of readers, with different ultimate aims. The two authors have different mathematical temperaments, their tastes differ. Professor Murnaghan has kept the modern algebraic view more or less in the background, and sometimes is less insistent on general results than, perhaps, the algebraist would like to see. This should not be understood as a criticism of the book. The avowed aim of the author was to make the theory of representations of the most important groups accessible in a relatively elementary way, and to write a self-contained book, without borrowing much from other theories. It is not the connection of the theory with other parts of algebra and mathematics in which Murnaghan is interested. The need to study the representations of certain groups has arisen, for instance in physics, and he wants to find out as much as possible about these representations, in the most direct way.

Professor Murnaghan's book is written with the clarity which could be expected from the author. All the details are worked out with great care. So the book should prove of high value to mathematicians and physicists who want to become acquainted with the material. Very little special knowledge is required of the reader. The volume can be warmly recommended.

There are a number of misprints, but hardly any of them are of a serious nature. Professor Murnaghan authorizes me to say that a list of misprints is available and will be supplied on request.

RICHARD BRAUER

ANNOUNCEMENT OF THE PROJECT FOR THE COMPUTATION OF MATHEMATICAL TABLES

FEDERAL WORKS AGENCY
WORK PROJECTS ADMINISTRATION
NEW YORK, N. Y.

Sponsored by the Bureau of Standards, Washington, D. C., Dr. Lyman J. Briggs, Director, and Dr. Arnold N. Lowan, Technical Director of the Project. Administrative Staff: Mr. Murray Pfeferman and Mr. Milton Abramowitz. Technical Staff: Dr. Gertrude Blanch, Dr. Norman Davids, Mr. William Horenstein, Mr. Jacob L. Miller, Miss Ida Rhodes; Dr. J. Rosenthal, and Mr. Herbert E. Salzer.

The Project for the Computation of Mathematical Tables has been in operation since January 1, 1938. The aim of this project is to compute Mathematical Tables of fundamental importance in Mathematics, Physics, Chemistry, Engineering, Statistics and related sciences. Lists of (A) Tables Published, (B) Tables in Process of Reproduction, (C) Tables for Which Manuscripts are Completed, (D) Tables for Which Computations are Completed, (E) Tables for Which Computations are in Progress and (F) Tables under Consideration, are given in this announcement.

The project would welcome suggestions for the computation of tables of interest in pure and applied Mathematics as well as information regarding computational work in progress elsewhere. Communications should be addressed to Oliver A. Gottschalk, Acting Administrator, Work Projects Administration, 70 Columbus Avenue, New York City.

Requests for copies of published tables should be addressed to Dr. Lyman J. Briggs, Director, National Bureau of Standards, Washington, D. C.

A. Tables Published

1. Table of the First Ten Powers of the Integers from 1 to 1,000. 80 pp. (1939).
2. Tables of the Exponential Function e^x. 535 pp. (1939).

$x = [-2.5000(0.0001)1.0000;\ 18D]$,[1] $x = [1.0000(0.0001)2.5000;\ 15D]$,
$x = [2.500(0.001)5.000;\ 15D]$, $x = [5.00(0.01)10.00;\ 12D]$.

3. Tables of sin x and cos x for Radian Arguments. 275 pp. (1940).

$x = [0.000(0.001)25.000;\ 8D]$, $x = [0(1)100;\ 8D]$,
$x = [0.00001(0.00001)0.00009;\ 15D]$, $x = [0.0001(0.0001)0.0009;\ 15D]$,
$x = [0.001(0.001)0.009;\ 15D]$, $x = [0.01(0.01)0.09;\ 15D]$,
$x = [0.1(0.1)0.9;\ 15D]$, $x = [0.00000(0.00001)0.01000;\ 12D]$.

There is also included a conversion table between radians and degrees.

4. Tables of sin x, cos x, sinh x, cosh x for Radian Arguments. 405 pp. (1940).

$x = [0.0000(0.0001)1.9999;\ 9D]$, $x = [0.0(0.1)10.0;\ 9D]$.

There is also included a conversion table between radians and degrees.

5. Tables of Planck's Radiation and Photon Functions. Published in the Journal of the Optical Society of America, vol. 30, pp. 70–81 (1940).

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[1] The figures in the square brackets give the range and the interval of the argument, and the number of decimal places (D) or significant figures (S) in the entries.

6. Table of $Si(x)$, $Ci(x)$, $Ei(x)$ and $-Ei(-x)$, where

$$Si(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad Ci(x) = \int_\infty^x \frac{\cos t}{t}\,dt,$$

$$Ei(x) = \int_{-\infty}^x \frac{e^t}{t}\,dt, \qquad -Ei(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt,$$

$$x = [0.0000(0.0001)2.0000;\ 9D].$$

7. Table of $Si(x)$, $Ci(x)$, $Ei(x)$ and $-Ei(-x)$,

$$x = [0.000(0.001)10.000;\ 9S,\ 10S]_1.$$[1]

8. Tables of Grid Coordinates (American Polyconic Projection) at 5 minute intervals of latitude and longitude for latitudes from 7° to 28°. (For the War Department.)

B. Tables in Process of Reproduction

1. Table of $\log_e N$, $N = [1(1)100{,}000;\ 16D]$.
2. Table of $\log_e x$, $x = [0.0000(0.0001)10.0000;\ 16D]$.
3. Table of the Probability Functions:

$$\frac{2}{\pi^{1/2}}e^{-x^2}, \qquad \frac{2}{\pi^{1/2}}\int_0^x e^{-t^2}dt,$$

$$[x = 0.0000(0.0001)1.0000;\ 15D], \qquad x = [1.000(0.001)5.736;\ 15D].$$

4. Table of the Probability Functions:

$$\frac{1}{(2\pi)^{1/2}}e^{-x^2/2}, \qquad \frac{1}{(2\pi)^{1/2}}\int_{-x}^x e^{-t^2/2}dt,$$

$$x = [0.0000(0.0001)1.0000;\ 15D], \qquad x = [1.000(0.001)7.800;\ 15D].$$

5. Tables of Section Moduli and Moments of Inertia for Structural Members used in Naval Architecture. (For the Bureau of Marine Inspection and Navigation.)
6. Table of the Electronic Functions

$$G = \frac{1}{(1-\beta^2)^{1/2}}, \qquad V = \frac{m_0c^2}{e}(G-1) \quad \text{and} \quad H\rho = \frac{m_0c^2}{e}\beta G$$

for β ranging from 0 to 0.99999 99990 at various intervals.
7. Table of arc tan x for

$$x = [0.000(0.001)7.000(0.01)50.00(0.1)300.0(1)2000(10)10000;\ 12D].$$

8. Table of Five-Point Lagrangian Interpolants for arguments ranging between 0 and 2, intervals of 0.001. (For the War Department.)

C. Tables for which Manuscripts are Completed

1. Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for complex arguments, $z = \rho e^{i\theta}$, for ρ ranging from 0 to 10 at intervals of 0.01 and θ ranging from 0° to 90° at intervals of 5°, to 10 decimal places.
2. Table of the definite integrals:

$$A(k, n) = \int_0^1 x^k \sin n\pi x\,dx \quad \text{and} \quad B(k, n) = \int_0^1 x^k \cos n\pi x\,dx \quad \text{for} \quad k = 0,$$

$$1, 2, \cdots, 10 \quad \text{and} \quad n = 1, 2, \cdots, 100 \text{ to 15 decimals.}$$

3. Table of $\int_0^x J_0(t)dt$ from 0 to 10 at intervals of 0.01 to 10 places.

4. Table of exact values of five-point Lagrangian Interpolants. (For the War Department.)

D. Tables for which Computations are Completed

1. Table of the fractional powers x^y for both x and y ranging from 0 to 1 at intervals of 0.01 and for other values of x and y to 15 decimal places.

2. Tables of Stellar Functions for "Point-Source" Models.

E. Tables for which Computations are in Progress

1. Bessel Functions of the fractional orders $\pm 1/3$, $\pm 2/3$, $\pm 1/4$, $\pm 3/4$ for real and pure imaginary arguments ranging from 0 to 10 at intervals of 0.01 to 10 decimal places.

2. Table of $Q_n(x) = (\pi/2x)^{1/2} J_{n+1/2}(x)$ for $n = 1, 2, 3, \cdots, 20$ and x ranging from 0 to 10 at intervals of 0.01 to 10 decimal places.

3. Table of the Bessel Functions $Y_0(z)$ and $Y_1(z)$ for the same complex arguments as in *C*.1.

4. Tables of tan x and cot x for x ranging from 0 to 2 radians at intervals of 0.0001.

5. Table of $Q_n(x) = (\pi/2x)^{1/2} J_{n+1/2}(x)$ for $n = -21, -20, \cdots, 0$ and x ranging from 0 to 10 at intervals of 0.01 to 10 decimal places.

6. Tables of $Si(x)$ and $Ci(x)$ for x ranging from 10 to 100 at intervals of 0.01.

7. Various hydraulic tables based on Kutter's and Manning's formulae. (For the War Department.)

8. Table of the Associated Legendre Functions of the First and Second Kind, for n ranging between 1 and 10, and m between 0 and 4; for arguments x and ix where x ranges between 0 and 10 at intervals of 0.1. Also corresponding values for half-integral values of n, and values of the functions for arguments in degrees. (Tabulation suggested by the National Defense Research Committee.)

9. Tables of Grid Coordinates of Northwestern Extension of U. S. (American Polyconic Projection. For the War Department.)

10. Table for Map Projections of Northwestern Extension of U. S. (American Polyconic Projection. For the War Department.)

11. Tables of Length of Meridional Arc at one-minute intervals. (For the War Department.)

F. Tables under Consideration

1. A 12 place table of Inverse Circular and Hyperbolic Functions other than arc tan x.

2. Extensive tables of Elliptic Integrals and Elliptic Functions.

3. Table of the Gamma Function for complex arguments, $x+iy$, for both x and y ranging from 0 to 5 at intervals of 0.05.

4. Table of the Error Functions for complex arguments.

5. The zeros of the Legendre Polynomials up to the 16th order to 15 decimal places and the weight coefficients for Gauss' Mechanical Quadrature Formula.

6. Tables of the Confluent Hypergeometric Functions for selected values of the parameters.

7. Table of the integral $\int_0^x Y_0(t)dt$.

8. Tables of three-point, four-point, and six-point Lagrangian Interpolants.

9. Table of the first ten powers of the reciprocals of the integers from 1 to 1000.

10. Table of solutions of Mathieu's Differential Equation.

NOTES

There is a mistake in the naming of the Committees to Select Hour Speakers on page 179 of the March Bulletin. These committees are: For the Annual and Summer Meetings, Professors J. R. Kline, Chairman, L. M. Graves and M. H. Ingraham; For Eastern Sectional Meetings, Professors T. R. Hollcroft, Chairman, C. R. Adams and Einar Hille. Those for the Western and Far Western Sectional Meetings are correct as announced.

The Mathematical Association of America plans a series of publications called The Herbert Ellsworth Slaught Memorial Papers. These papers will be primarily of expository character, probably between forty and eighty printed pages in length.

The Sixth International Congress for the Unity of Science will be held at the University of Chicago September 2–6, 1941.

A symposium including an address by Professor G. A. Bliss of the University of Chicago on *Functions of lines in ballistics* and an address by Professor Marston Morse of the Institute for Advanced Study on *Some new methods and problems in minimal surface theory* was held at the University of Cincinnati on November 9, 1940.

A symposium on mathematical economics was held at the University of Notre Dame February 28 and March 1, 1941. The following addresses were given: *Functionals in the theory of economics*, Professor H. T. Davis, Northwestern University; *The theory of technological unemployment*, Professor Oscar Lange, University of Chicago; *Problems of valuation and rate-making, in public enterprises*, Professor Harold Hotelling, Columbia University; *The dynamics of commodity prices*, Dr. C. F. Roos, Institute for Applied Economics, New York City; *Statistical cost-curves and price policy*, Professor Joel Dean, University of Chicago; *The determination of weight-functions for dynamic economics*, Dr. Franz Alt, Institute for Applied Economics; *Unexpected effects of certain taxes*, Professor Harold Hotelling; *On the solvability of the Walrasian system of equations*, Dr. Abraham Wald, Columbia University; *The geometrical and arithmetical determination of the index of prices*, Professor Karl Menger, University of Notre Dame.

Professor Lawrence Crawford of South African College, Cape Town, has been elected president of the Royal Society of South Africa.

At the Annual Meeting of the Mathematical Association of America Professor R. W. Brink of the University of Minnesota was elected president, Professor R. E. Langer of the University of Wisconsin first vice president, and Professor B. H. Brown of Dartmouth College second vice president.

Professor Einar Hille of Yale University has been elected a member of the Royal Physiographical Society of Lund, Sweden, as successor of Professor Vito Volterra.

Associate Professor A. A. Albert of the University of Chicago delivered the William Lowell Putnam lecture at Harvard University in March. He spoke also at Princeton University, the Institute for Advanced Study and Brown University.

The following eighty-nine doctorates, with mathematics or mathematical physics as a major subject, were conferred during 1940 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics), and the title of the dissertation are given in each case if available.

P. H. Anderson, Illinois, June, minor in economics, *Distributions in stratified sampling.*

L. A. Aroian, Michigan, February, *A study of R. A. Fisher's t-distribution and the related F-distribution.*

Fred Assadourian, New York, June, minor in physics, *Intransitive Abelian almost-translation groups of almost-periodic functions.*

A. A. Aucoin, Louisiana State, May, *An algebraic treatment of diophantine analysis.*

R. A. Beaumont, Illinois, June, minor in astronomy, *Projections of non-Abelian groups upon Abelian groups containing elements of infinite order.*

E. G. Begle, Princeton, June, *Locally connected spaces and generalized manifolds.*

J. O. Blumberg, Pittsburgh, August, *Fundamental regions in S_6 for the simple quaternary $G^4_{60\phi}$, Type* II.

J. I. Bohnert, Pittsburgh, August, *Conical surface roulettes.*

A. A. F. Brown, Princeton, June, *Jordan fields on groups.*

G. W. Brown, Princeton, April, *The reduction of a certain class of composite statistical hypotheses.*

R. H. Bruck, Toronto, June, *The general linear group in a field of characteristic p.*

E. F. Canaday, Duke, June, *The sum of the divisors of a polynomial.*

R. F. Clippinger, Harvard, June, *Matrix products of matrix powers.*

R. H. Cole, Wisconsin, June, *The expansion problem associated with an ordinary linear differential equation and boundary conditions applying at a set of colinear points.*

R. H. Cook, Iowa State, December, major in applied mathematics, minors in physics and mathematical physics, *Evaluation of error in approximation by function methods.*

W. E. Cox, Ohio State, June, *On commutative normal matrices and unitary equivalence of matrices.*

A. B. Cunningham, West Virginia, June, minor in physics, *Non-involutorial space transformations associated with a $Q_{1,2}$ congruence.*

Norman Davids, New York, June, minor in physics, *Minimal surfaces spanning closed manifolds and having prescribed topological position.*

Bernard Dimsdale, Minnesota, July, minor in physics, *Lacunary orthogonal polynomials.*

W. D. Duthie, Princeton, June, *Segments in ordered sets.*

John Dyer-Bennet, Harvard, June, *A free algebra with three operations.*

C. J. Everett, Wisconsin, March, *Rings as groups with operators.*

Sheng-Chin Fan, Michigan, June, *Integration with respect to an upper measure function.*

H. T. Fleddermann, Louisiana State, May, *Density properties of sets.*

Sister Elizabeth Frisch, Catholic University, June, minors in physics and biology, *Determination of a set of independent relations characterizing a certain system consisting of a conic and quartic curve.*

Holly C. Fryer, Iowa State, July, major in mathematical statistics, minors in mathematics and genetics, *An analysis of group differences arising from a Poisson distribution of observations obtained from irradiation experiments.*

R. E. Gaskell, Michigan, June, *A problem in heat conduction and an expansion theorem.*

A. M. Gelbart, Massachusetts Institute of Technology, June, minor in physics, *Growth properties of functions of two complex variables.*

J. H. Giese, Princeton, June, *Conformally flat hypersurfaces of spaces of constant curvature.*

P. W. Gilbert, Duke, June, minors in physics and philosophy, *N-to-one mappings of linear graphs.*

B. E. Gillam, Missouri, June, *A new set of postulates for euclidean geometry.*

H. E. Goheen, Stanford, June, minor in physics, *On the primitive groups of classes* 4P *and* 5P.

Katharine E. Hazard, Chicago, August, *Index theorems for the problem of Bolza in the calculus of variations.*

M. H. Heins, Harvard, June, *Studies in the theory of functions analytic and single-valued in a multiply-connected region.*

A. D. Hestenes, Harvard, June, *On the transformation* $x'^{-c}e^{x'} = x^{-c}e^{x+1}$ *and the related linear* ϕ*-difference equations of G. D. Birkhoff.*

J. F. Heyda, Illinois, June, minor in physics, *Uniqueness properties over sets of positive linear measure of functions of a complex variable.*

F. B. Hildebrand, Massachusetts Institute of Technology, June, minor in physics, *Solution by polynomial approximation of singular integral equations arising in static field theory.*

L. C. Hutchinson, Massachusetts Institute of Technology, June, minor in physics, *On the classification of the trivector.*

R. F. Jackson, Harvard, June, major in mathematical physics, *Reciprocity theorems in the theory of atomic spectra.*

F. P. Jenks, Notre Dame, June, *A new set of postulates for Bolyai-Lobachevsky geometry.*

J. L. Kelley, Virginia, June, *A study of hyperspaces.*

Fred Kiokemeister, Wisconsin, June, *The parastrophic criterion for the factorization of primes.*

H. L. Langhaar, Lehigh, June, *Steady flow in the transition length of a cylindrical conduit.*

J. H. Levin, Chicago, August, *Minima of double integrals with respect to unilateral variations, and applications to subharmonic functions.*

P. E. Lewis, Illinois, June, minor in physics, *Characters of Abelian groups.*

D. T. McClay, Harvard, June, *Analytic families of somas.*

A. W. McGaughey, Cincinnati, June, *Lacunary double Fourier series.*

Ingo Maddaus, Michigan, February, *Types of "weak" convergence of linear normed spaces.*

Dorothy Maharam, Bryn Mawr, June, *The notion of measure on abstract sets.*

A. V. Martin, Duke, June, minor in philosophy, *Monotone transformations of non-compact two-dimensional manifolds.*

A. B. Mewborn, California Institute of Technology, June, minor in physics, *Contributions to the general geometry of paths.*

C. E. Miller, Toronto, June, *On finite collineation groups of a given degree.*

W. L. Mitchell, Wisconsin, July, *Topological rings and infinite matrices.*

J. C. Montgomery, Yale, June, *Relations between the roots of a polynomial and the roots of its first derivative.*

A. M. Mood, Princeton, June, *The distribution theory of runs.*

C. W. Moran, Illinois, June, minor in physics, *Asymptotic theory of linear differential equations singular in several parameters.*

W. L. Morris, Duke, June, minor in physics, *Kinetic-harmonic systems.*

H. T. Muhly, Johns Hopkins, June, *Valuations and infinitely near algebraic loci.*

C. G. Mumford, Duke, June, *Topics in double Fourier series.*

J. P. Nash, Rice Institute, June, *A class of continuous functions and convergence criteria for their Fourier series.*

Albert Neuhaus, Chicago, March, *Products of normal semi-fields.*

H. E. Newell, Wisconsin, June, *On the asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain.*

K. L. Nielsen, Illinois, June, minor in astronomy, *General boundary value problems for linear differential equations.*

J. M. H. Olmsted, Princeton, June, *Lebesgue theory on a Boolean algebra.*

R. S. Pate, Illinois, June, minor in astronomy, *Systems with operations which are not single-valued.*

H. V. Price, Iowa, August, minors in applied mathematics and education, *Analyses of a group of pre-tests for students of first year college mathematics.*

M. O. Reade, Rice Institute, June, *Generalizations to space of the Cauchy and Morera theorems.*

C. J. Rees, Pennsylvania, February, *Elliptic orthogonal polynomials.*

F. D. Rigby, Iowa, August, minors in applied mathematics and physics, *Topologies in distributive lattices.*

C. V. Robinson, Missouri, June, *Contributions to distance geometry:* (1) *Congruence order of euclidean and spherical subsets.* (2) *Helly theorems on the sphere.*

N. N. Royall, Brown, June, *On Laplace transforms of multiply monotonic functions.*

J. M. Sachs, Chicago, December, *Theory of surfaces in conjugate parameters.*

W. J. Schart, Ohio State, August, *Conditions for solutions of certain differential equations which have specified properties.*

Edith R. Schneckenburger, Michigan, February, *On 1-bounding monotonic transformations which are equivalent to homeomorphisms.*

K. C. Schraut, Cincinnati, June, *Convergence factor theorems for double and multiple series having unbounded partial sums.*

I. E. Segal, Yale, June, *Ring properties of certain classes of functions.*

C. E. Shannon, Massachusetts Institute of Technology, June, minor in electrical engineering, *Algebra for theoretical genetics.*

R. W. Shephard, California (Berkeley), August, I. *Incidence of taxation studied in a simplified economic system.* II. *A note on the length of production.*

Seymour Sherman, Cornell, June, minor in philosophy, *A comparison of linear measures in the plane.*

S. S. Smith, Chicago, August, *Motion of a spheroid and a sphere under mutual gravitation.*

Abraham Spitzbart, Harvard, June, *Approximation in the sense of least pth powers by polynomials with auxiliary conditions of interpolation.*

D. W. Starr, Illinois, June, minor in political science, *The Schrödinger wave equation from the point of view of singular integral equations.*

B. M. Stewart, Wisconsin, July, *Left-associated matrices with elements in an algebraic domain.*

G. B. Thomas, Cornell, June, *Regular ternary quadratic forms.*

V. J. Varino, Wisconsin, July, *An extension of the theory of matrices with elements in a principal ideal ring.*

L. F. Walton, California (Berkeley), August, *On ideal numbers.*

J. S. de Wet, Princeton, June, *On the connection between the spin and statistics of elementary particles.*

A. L. Whiteman, Pennsylvania, February, *Additive prime number theory in real quadratic fields.*

Orla V. Wood, California (Berkeley), May, *On the linear homogeneous differential equation of the second order with four regular singular points.*

Max Wyman, California Institute of Technology, June, minor in physics, *General differential geometry with two types of linear connection.*

The following doctorates were conferred in 1939, but were not included in the list in the preceding volume of this Bulletin (vol. 46, pp. 391–397).

J. F. Daly, Princeton, October, *On the unbiased character of likelihood-ratio tests for independence in normal systems.*

G. N. Garrison, Princeton, October, *Quasi-groups.*

Andrew Sobczyk, Princeton, October, *Projections in Minkowski and Banach spaces.*

M. E. Wescott, Northwestern, August, minor in economics, *A set of Newton polynomials analogous to Laguerre's polynomials.*

G. W. Whaples, Wisconsin, *On the structure of modulus with a commutative algebra as operator domain.*

Assistant Professor J. E. Eaton of Hofstra College has been promoted to an associate professorship.

Dr. K. C. Schraut of the University of Dayton has been promoted to an assistant professorship.

Assistant Professor Saunders MacLane of Harvard University is on leave this term at the University of Chicago.

The following appointments to instructorships are announced: Cornell University: Mr. W. H. Durfee; Fenn College (Cleveland): Mr. C. W. Topp; Harvard University: Dr. I. E. Segal; University of Michigan: Dr. Sam Perlis.

Mr. H. W. Macrosty, president of the Royal Statistical Society, died January 19, 1941, at the age of seventy-six years.

Professor Francesco Palatini of the Technical Institute of Turin died June 2, 1940, at the age of seventy-four years.

Professor Emeritus E. S. Loomis of Baldwin-Wallace College died December 11, 1940.

Professor Emeritus D. C. Miller of Case School of Applied Science died February 22, 1941, at the age of seventy-four years. He was a member of the Society from 1923 to 1936.

The death of Mr. J. O. Watts of Queen's University in Kingston, Ontario, has been announced.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

182. Reinhold Baer: *A unified theory of projective spaces and finite abelian groups.*

It is the object of this investigation to develop in detail a theory which contains as special cases both projective geometry and the theory of finite abelian groups. The primary abelian operator groups lend themselves in quite a natural manner for this purpose. Those partially ordered sets that are exactly the systems of all the admissible subgroups of suitable primary abelian operator groups are determined, and it is shown that the group and its operators are completely determined by the system of admissible subgroups. Those questions which are fundamental in the two theories are discussed, such as the basis-theorem, the relations between dualities and bilinear forms, and so on. (Received March 12, 1941.)

183. Richard Brauer: *On the connection between the ordinary and the modular characters of groups of finite order.*

Let G be a group of finite order $g=p^ag'$, where p is a fixed prime and $(p, g')=1$. If $\mathfrak{Z}$ is a character of G, then for every element A of an order prime to p the value $\mathfrak{Z}(A)$ is a linear combination of the modular group characters of G. The coefficients are rational integers, the decomposition numbers of G. In this paper it is shown that the value $\mathfrak{Z}(A)$ of $\mathfrak{Z}$ for elements A of an order divisible by p can be expressed by means of the modular characters of certain subgroups N_i of G. The matrix Z of all ordinary group characters of G then appears as a product DX of two square matrices. The matrix X contains in its rows the values of the modular characters of G and of the N_i while the coefficients of D are integers of the field of the p^ath roots of unity. A number of properties of the coefficients of D are given. (Received March 18, 1941.)

184. R. H. Bruck and T. L. Wade: *Bisymmetric tensor algebra.* I.

This paper lays a basis for a study of the linear associative algebra of bisymmetric tensors (H. Weyl, *The Classical Groups*, p. 98) which gives a realization of a semi-simple algebra. The ordered product of two tensors $A^{(i)}_{(j)}$ and $B^{(i)}_{(j)}$ is defined by $A^{(i)}_{(m)}B^{(m)}_{(j)}=C^{(i)}_{(j)}$, where $(i)=i_1\cdots i_p$. The unit tensor $\delta^{(i)}_{(j)}=\delta^{i_1}_{j_1}\cdots\delta^{i_p}_{j_p}$ may be decomposed into a direct sum of immanent tensors (T. L. Wade, abstract 47-1-10). For a tensor $A^{(i)}_{(j)}$, a determinant, adjoint, and inverse are defined with the aid of a tensor $\delta^{(i_1)\cdots(i_N)}_{(j_1)\cdots(j_N)}$ which constitutes a further generalization of the familiar generalized Kronecker delta; here $N=n^p$. Also the concepts of rank and of rank tensor are introduced. Explicit algebraic construction is given of the factors of the determinant in the

case of a bisymmetric tensor. The determinant of a Kronecker product (Murnaghan, *The Theory of Group Representations*, pp. 68–69) is a special case of the determinant defined in this paper. (Received March 27, 1941.)

185. R. H. Bruck and T. L. Wade: *Bisymmetric tensor algebra. II.*

If an idempotent numerical tensor $C_{\omega}^{(\omega)}$ has rank r (as defined in part I), its rank tensor, an absolute numerical tensor, factors into the product of two relative tensors. For a certain invariant subalgebra of the algebra of bisymmetric tensors, $C_{\omega}^{(\omega)}$ is the unit, and for this subalgebra the rank tensor plays the role of the generalized Kronecker delta for $p=1$ (see part I). $C_{\omega}^{(\omega)}$ may be thought of as the idempotent numerical tensor corresponding to an operator c, or as an immanent tensor corresponding to an ϵ, where c and ϵ are used in the sense of H. Weyl (*The Classical Groups*, pp. 120–127). In either case a determinant, adjoint, and inverse are defined, and the relation between the two cases is made explicit. In connection with this work the homomorphic correspondence between the group-ring of the symmetric group on p letters and the algebra of absolute numerical tensors is established and used considerably. (Received March 27, 1941).

186. D. M. Dribin: *A theorem on sets of ideals in solvable extensions of algebraic number fields.*

The following theorem is proved: Let k be an algebraic number field and $K|k$ a normal extension with solvable group. Let Π be any set of prime ideals in k which decompose into prime ideals of relative degree one in K and which are all equivalent in k in the ordinary (broad) sense. Then if Π_K is the set of all prime ideal divisors in K of the ideals in Π, all the ideals of Π_K are equivalent (in the ordinary sense) in K. (Received March 14, 1941.)

187. Fred Kiokemeister: *The parastrophic criterion for the factorization of primes.*

The parastrophic form $\pi(\xi)$ of a Frobenius algebra is defined as the determinant of the parastrophic matrix in a given basis of the algebra. In particular the parastrophic form may be computed with respect to the basis of a maximal domain of integrity K of an algebraic number field. Let p be a rational prime number. Then the prime ideal factorization $(p)=\mathfrak{p}_1^{\alpha_1}\mathfrak{p}_2^{\alpha_2}\cdots\mathfrak{p}_s^{\alpha_s}$ of p in K is paralleled by the factorization of $\pi(\xi)$ modulo p into a product of powers of irreducible factors. This result is obtained through a study of the relationship between the ideal structure of a linear algebra and the zeros of $\pi(\xi)$. (Received February 14, 1941.)

188. C. C. MacDuffee: *Products and norms of ideals.*

In an integral domain of a Frobenius algebra over the quotient field of a principal ideal ring, the number h of ideal classes is finite. To every left (or right) ideal $\mathfrak{a}$ correspond h matrices $A_1, A_2, \cdots, A_h$, one for every ideal class of the domain. The first matrix A_1, corresponding to the principal class, is a matrix which determines a minimal basis of $\mathfrak{a}$. If $B_1, B_2, \cdots, B_h$ correspond similarly to the ideal $\mathfrak{b}$ which belongs to the ith ideal class, then the matrix $C_1=A_iB_1$ corresponds to the product ideal $\mathfrak{a}\times\mathfrak{b}$. Thus ideal multiplication can always be accomplished by means of matric multiplication. Call $|A_i|=N_i(\mathfrak{a})$ the ith norm of $\mathfrak{a}$, and $N_1(\mathfrak{a})$ the principal norm. It is known that the principal norm of a product is not always equal to the product of the principal norms. It is here proved, however, that the principal norm of $\mathfrak{a}\times\mathfrak{b}$ is always equal to

the product of the ith norm of $\mathfrak{a}$ by the principal norm of $\mathfrak{b}$. (Received March 31, 1941.)

189. Saunders MacLane and O. F. G. Schilling: *A formula for the direct product of cross product algebras.*

Let N be a separable normal extension of a field L with Galois group Γ. Suppose that K is a normal subfield of N belonging to the group Δ. The authors prove the formula $(N, \Gamma, F)\times(K, \Gamma/\Delta, G)=(N, \Gamma, FG^*)$ where F, G are factor sets belonging to N, K, respectively. The factor set G^* of N is obtained by "Δ-symmetric extension" of G to N. (Received March 20, 1941.)

190. Saunders MacLane and O. F. G. Schilling: *Group extensions characterizing complete fields.*

Let F be a field which is relatively complete with respect to a valuation Φ with residue class field $\mathfrak{f}$. The authors define the "dominant" F' of F as the factor group of the multiplicative group of F modulo the group of units u with $\Phi(1-u)>0$. The group F' is a group extension of the multiplicative group of $\mathfrak{f}$ by the value group. Suppose that K is a finite separable extension of F whose degree over the unramified field is prime to the characteristic of F. The dominant K' of K is then a "linked" extension of F'. The authors prove that the extensions $K\supset F$ are in 1-1 correspondence with the linked extensions $K'\supset F'$. The type number defined by Albert (Annals of Mathematics, (2), vol. 41 (1940), pp. 678–679) is essentially an integer describing a cyclic extension $K'\supset F'$. (Received March 20, 1941.)

191. Saunders MacLane and O. F. G. Schilling: *The principal divisor theorem for function fields.*

Let F be a field of algebraic functions of one variable. The authors investigate the structure of extensions $K\supset F$ in which the divisors of a given class group B of F become principal. An unramified extension $K\supset F$ in which exactly the divisors of B become principal is termed a Hilbert class field for B. In general there exist several "minimal" normal Hilbert class fields belonging to a group B. The Galois groups of such separable normal fields are abelian if an only if (i) the order of B is prime to the characteristic of F, and (ii) the characters of B can be realized in F. The methods of proof differ radically from the ones used in the theory of algebraic number fields. (Received March 20, 1941.)

192. I. M. Niven: *Equations in quaternions.*

It is proved that the equation $x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=0$, with real quaternion coefficients, has a root in the algebra of real quaternions when n is odd. Also, the number of roots of this equation may be finite or infinite, and if finite, does not exceed $(2n-1)^2$. (Received March 20, 1941.)

193. I. M. Niven: *Sums of nth powers of quadratic integers.*

The first problem is to find those quadratic fields all of whose integers are expressible as sums of nth powers. This is solved completely, criteria being given in terms of n and the basis of each field. The second problem is to determine whether any given quadratic integer is expressible as a sum of nth powers. Necessary and sufficient conditions are again given; in this case however, integers of real quadratic fields are not treated when n is even. (Received March 5, 1941.)

194. R. S. Pate: *Rings with multiple-valued operations.*

A set R of elements forming a hypergroup with respect to addition and having a distributive associative multiplication such that ab is a subset of elements of R is defined to be a ring. If R is an additive group every product contains the same number n of elements, and n is a divisor of the order of the group. A subring A of R such that $AR \subset A$ is an ideal. To a fixed element q, to every element r of R may be associated an arbitrary set of corresponds x_i and y_i such that $r+x_i \supset q$ and $r \subset y_i+q$. A Q ideal contains q, and, if it contains r, contains all corresponds of r. Usually, the necessary and sufficient condition that a set of ideals form a lattice is that they be Q ideals. For a proper correspondence Q, the classical decomposition of ideals may be derived. An analogue of an indeterminate domain may be defined and a result derived similar to the Gauss lemma. It can be shown that the "indeterminate domain" of a ring with a "well-behaved" multiplication has this same multiplication only if it is an ordinary ring. (Received March 17, 1941.)

195. Sam Perlis: *A characterization of the radical of an algebra.*

The following theorem is proved: If F is any field and A is an algebra over F with a unity element, the radical of A consists of all elements h such that $g+h$ is regular for every regular g. If A does not have a unity element, one may be adjoined without altering the radical. (Received March 14, 1941.)

196. Everett Pitcher and M. F. Smiley: *Transitivities of betweenness.*

The importance of the transitivity "abc, adc, and $bxd \rightarrow axc$" in the theories of lattices and of metric spaces leads the authors to investigate all possible five point transitivities of betweenness on a line whose conclusions state that *just one* relation holds and whose hypotheses consist of three relations. Thirty-three distinct postulates are found. Of these, only ten fail to be equivalent to combinations of the fundamental properties (1), (2), and (3) of Huntington and Kline (Transactions of this Society, vol. 18 (1917), p. 305). A discussion of the two possible weak transitivities on four points is included. The influence of each of these transitivities, as well as the fundamental ones of Huntington and Kline, when applied to a definition of betweenness in lattices (this Bulletin, abstract 47-5-201) is investigated. Eight of the five point transitivities are shown to hold in metric ptolemaic spaces, while only one of them is valid in every metric space. (Received March 31, 1941.)

197. H. J. Riblet: *Certain theorems for symmetric differential functions.*

Symmetric differential functions, in which the n quantities $y_1, \cdots, y_n$ are not all distinct, are considered. It is shown that the symmetric function theorem still holds although in different form. For the case in which $y_1, \cdots, y_n$ are all distinct, an improved bound is given for the power of the discriminant which occurs in the expression of any symmetric differential function in terms of the elementary symmetric functions and their derivatives. (Received March 31, 1941.)

198. W. M. Scott: *On matrix algebras over an algebraically closed field.*

The matrix algebra A is considered as an (A, A) module, that is to say, as an

abelian group with A as a right and left operator system. By taking A in reduced form, a set of parts C_{ij} is obtained, of which the C_{ii} are the irreducible constituents of A. These parts C_{ij} are called the simple parts of A, as each forms a simple (A, A) module. Each nonzero simple part is isomorphic to a composition factor group of the module A. A second set of simple (A, A) modules, the elementary modules of A, is defined by use of the Cartan basis system employed by C. Nesbitt in his study of the regular representations of algebras (Annals of Mathematics, (2), vol. 39 (1938), pp. 634–658). A (1-1) correspondence exists between the elementary modules and the composition factor groups of A. The simple parts of A are expressible as linear combinations of these elementary modules, and a similar statement may be made concerning simple parts of representations of A. A classification of the simple parts of A, by use of chains, gives a decomposition of A by elementary transformations into parts, each of which defines a directly indecomposable invariant subalgebra of A. (Received March 25, 1941.)

199. H. A. Simmons: *Classes of maximum numbers associated with symmetric equations in n reciprocals.* IV.

The equation considered here is $\sum_{i=1}^{n}(1/x_i)+\sum_{i=1}^{m}a_i(\prod(x))^{-i}=b/a$, $a\equiv(c+1)^b-1$, in which n is an integer >1, b, c, m are arbitrary positive integers, $\prod(x)=x_1x_2\cdots x_n$, and the a_i are non-negative integers. Using the terminology of the author's article with W. E. Block (Duke Mathematical Journal, vol. 2, p. 317), the results of the present paper are as follows: The largest product of the numbers in any E-solution of the equation is the product of the w_i of the Kellogg solution w of the equation; this product is not attained by the numbers in any E-solution except w. With certain restrictions on the a_i ($i=2, 3, \cdots, m$), the Kellogg solution has the two remarkable properties that were established for the Kellogg solutions employed in the article referred to above. (Received March 12, 1941.)

200. H. A. Simmons: *Use of matrices in solving linear Diophantine equations.*

Given the equation $\sum_{i=1}^{n}a_ix_i=k$, $n>1$, in which the a_i are integers different from zero and k is an integer. It is known that this equation has a solution (in integers) if and only if $(a_1, a_2, \cdots, a_n)$ divides k. The purpose of this paper is to express in matricial form the substitutions that are used in solving the equation by the method of substitutions (as given, say, in E. Cahen, *Théorie des Nombres*, vol. 1, chaps. 9, 10). Each matrix here used is nonsingular and every matrix employed except the last one has determinant ±1. A special order of carrying out the desired matric multiplications is advantageous. The method used in this paper is usually briefer than even the more refined recent methods that employ continued fractions. (Received March 12, 1941.)

201. M. F. Smiley: *Betweenness in general lattices.* Preliminary report.

V. Glivenko (*Contributions à l'étude des systèmes de choses normées*, American Journal of Mathematics, vol. 59 (1937), pp. 941–956, and *Géométrie des systèmes de choses normées*, American Journal of Mathematics, vol. 58 (1936), pp. 799–828) studied the relations between the properties of the lattice operations and metric betweenness in a metric lattice. He showed that c was metrically between a and b if and only if $(a\cap c)(b\cap c)=c=(a\cup c)(b\cup c)$. In particular, he characterized distributive lattices

as those in which betweenness is "transitive," that is, whenever c is between x and y and both x and y are between a and b, then c is between a and b. Starting with this criterion for betweenness as a definition to be applied to general lattices, this result is obtained without the use of a metric. It is also shown that the transitive property of Pasch which holds in any metric space, namely, b between a and c with c between a and d implies that b is between a and d (see K. Menger, *Untersuchungen über allgemeine Metrik*, Mathematische Annalen, vol. 100 (1928), pp. 78–80), characterizes modular lattices. (Received February 7, 1941.)

202. B. M. Stewart: *Left-associated matrices with elements in an algebraic domain.*

The n by n matrices A and B with elements in an algebraic domain of order k are said to be left associates if there exists a unimodular matrix P with elements in the domain such that $PA=B$. If the domain is a principal ideal ring, a necessary and sufficient condition that A and B be left associates is that A and B have the same Hermite normal form. For a domain whose class number is greater than one the presence of non-principal ideals prevents such a direct solution. But if each element of A is replaced by its k by k second matric representation there is produced an enlarged matrix A' of order kn by kn with elements in the rational domain, so that for A' the Hermite form is well-defined and easily found. This paper investigates the possibility that a necessary and sufficient condition that A and B be left associates is that the corresponding enlarged matrices A' and B' be left associates. The necessity is shown, and the sufficiency is proved for nonsingular matrices, for matrices whose column class is the principal class, and for matrices of rank 1. Use is made of the results of Steinitz concerning the equivalence problem $PAQ=B$. (Received March 8, 1941.)

203. H. P. Thielman: *Groups of linear fractional transformations.*

G. A. Miller (American Journal of Mathematics, vol. 22, p. 185) has shown that two operations s_1 and s_2, each of period two, can be selected in such a way as to give a product s_1s_2 of any desired period. In the present paper this theorem is applied to linear fractional transformations of the form $T\equiv(ax+b)/(cx+d)$. If T is of period two it must be of the form $S\equiv(ax+b)/(cx-a)$, $|a|+|c|\neq 0$. Two transformations $S_1\equiv(a_1x+b_1)/(c_1x-a_1)$ and $S_2\equiv(a_2x+b_2)/(c_2x-a_2)$ are selected so that the product will be of any given period n. The dihedral group is then formed by extending the cyclic group generated by S_1S_2 by one of the elements S_1 or S_2. If in S_1 $c_1=0$, and in S_2 $a_2=0$, the groups of subtraction and division are obtained (G. A. Miller, Quarterly Journal of Pure and Applied Mathematics, 1906, pp. 80–87; E. J. Finan, American Mathematical Monthly, 1941, pp. 3–7). Simple representations of all dihedral groups are obtained. The results are consequences of the functional equations satisfied by the coefficients of S_1 and S_2. (Received March 13, 1941.)

204. R. M. Thrall: *On Young's semi-normal representation of the symmetric group.*

The main body of the paper is devoted to a new (and shorter) derivation of Young's semi-normal orthogonal form for the irreducible representations of the symmetric group S_m on m letters. The procedure is such as to admit an immediate extension which yields a specific unitary form for the irreducible representations of the alternating group A_m. This new derivation is based on a definition of the semi-normal idempotents of the group ring R_m of S_m, directly in terms of the regular Young

diagrams ("standard tableaux") associated with the partitions of m. (Received March 11, 1941.)

205. G. L. Walker: *The elementary divisors of a direct product of matrices.*

If $A=(a_{ij})$ and $B=(b_{\alpha\beta})$ are square matrices, the direct product (or Kronecker product) of A and B is $A\otimes B=(a_{ij}B)$, that is, having elements $a_{ij}b_{\alpha\beta}$ arranged in rows by lexicographic order on i and α and in columns by lexicographic order on j and β. Known results on the characteristic roots of $A\otimes B$ are extended to the determination of the elementary divisors of $A\otimes B$ in terms of the elementary divisors of A and of B. The problem is first reduced to the case of A and B having the single elementary divisors, say $(\lambda-\rho)^m$ and $(\lambda-\sigma)^n$ respectively. Then $A\otimes B$ has elementary divisors: (i) $(\lambda-\rho\sigma)^{m-n-1+2t}$ where $t=1, 2, \cdots, n$ if $\rho\sigma\neq 0$ and $m\geqq n$; (ii) λ^m repeated n times if $\rho=0$, $\sigma\neq 0$; (iii) λ^k, λ^k where $k=1, 2, \cdots, n-1$ and λ^n repeated $m-n+1$ times if $\rho=\sigma=0$ and $m\geqq n$. (Received March 31, 1941.)

206. Morgan Ward: *Finite operators of formal power series.*

Let R be a ring of formal power series $A=\sum a_n x^n$ with coefficients in a commutative field. A finite operator ω transforms A into another series $\omega A=\sum a_n' x^n$ where a_n' is a function of $a_0, a_1, \cdots, a_{n-1}, a_n$ alone. The ring of all such operators is studied and in particular the subring of linear operators. The results include as special cases a previously developed "calculus of sequences" (Ward, American Journal of Mathematics, vol. 58 (1936), pp. 255–266), a large part of the symbolic calculus of Blissard, Lucas and Bell and Hurwitz' theory of integral power series. (Received March 8, 1941.)

207. Louis Weisner: *Polynomials whose roots lie in a sector.*

From two polynomials $A(z)=\sum a_k z^k$ and $B(z)=\sum b_k z^k$, form the composition-polynomials $f(z)=\sum a_k b_k z^k$ and $g(z)=\sum k! a_k b_k z^k$. The author proves that if the roots of $A(z)$ are real while those of $B(z)$ lie in a sector S with vertex at the origin and aperture less than or equal to π, then the roots of $f(z)$ and $g(z)$ lie in S or the sector vertical to S. When S is the positive or negative real axis, these results reduce to well known theorems of Malo and Schur. The author shows further that when the roots of $A(z)$ are negative, the roots of $f(z)$ and $g(z)$ lie in S. (Received March 13, 1941.)

208. L. R. Wilcox: *Extensions of semi-modular lattices.*

Let L be a semi-modular lattice in which $a, b\in L$ implies the existence of $c\in L$ with $a+c=a+b$, $(a, c)\perp$, and in which there are no points (L. R. Wilcox, Annals of Mathematics, (2), vol. 40 (1939), pp. 490–505). Restricted dual-ideals are sets $S\subset L$ such that $a\in S$, $b\geqq a$, implies $b\in S$, and $a, b\in S$, a not parallel to b, implies $ab\in S$. All restricted dual-ideals form a lattice L' in which L is order-isomorphically embedded. It is proved in this paper that the restricted dual-ideals determined by two-element sets $[b, c]$ form a complemented modular sublattice $\bar{L}$ of L', which contains the image of L. Properties of $\bar{L}$ are studied. (Received March 14, 1941.)

209. M. A. Woodbury: *On ordered groups.*

A group G is said to be ordered if it is simply ordered and the group operation preserves order. If the group G has the property that it contains an element u such

that for every $g \in G$ there exists a natural number n for which $u^n > g$, then G is homomorphic (preserves the group operation and $\leqq$) to a subgroup of the additive group of real numbers. If G has the further property that for any $g \in G$ preceded by the identity there is a natural number n for which $g^n > u$, then the homomorphism becomes an isomorphism. These two properties are equivalent to the Archimedean postulate. The Archimedean property is necessary for an isomorphism as is shown by an example of Reidemeister (*Grundlagen der Geometrie*, pp. 40–41) of a non-Archimedean ordered quasi-field. (Received March 14, 1941.)

210. Leonard Carlitz: *Some interpolation formulas connected with polynomials in $GF(p^n)$.*

The interpolation formulas discussed are of two types, namely, formulas arising from interpolation at $x^{p^{ni}}$, and secondly, interpolation at arbitrary $M(x)$ in $GF(p^n, x)$. This paper is closely connected with a previous paper (this Bulletin, abstract 47-1-80). (Received April 1, 1941.)

211. Leonard Carlitz: *The coefficients of the reciprocal of certain series.*

Given the "linear" function $f(t) = \sum_{i=0}^{\infty}(A_i/F_i)t^{p^{ni}}$, where the A_i are polynomials in $GF(p^n, x)$, define β_m by means of $t/f(t) = \sum_{m=0}^{\infty}(\beta_m/g_m)t^m$. In this paper various properties of β_m are discussed. Use is made of an identity of the form $f(xt) - xf(t) = \sum_{i=1}^{\infty}\alpha_i f^{p^{ni}}(t)$ and also of the inverse function of $f(t)$. (Received April 1, 1941.)

212. L. W. Griffiths: *The minimum number of variables in universal functions of polygonal numbers.*

The universal functions of polygonal numbers of order $m+2$, in which the sum of the coefficients is not greater than $m+2$, were determined in an earlier paper (Annals of Mathematics, (2), vol. 31 (1930), pp. 1–12). In each such universal function the number n of variables was at most $m+2$. In this paper a universal function is exhibited for each m which involves the smallest number of variables appearing in any of these universal functions for that m, and the value of this minimum n is determined in terms of m. (Received February 28, 1941.)

213. D. H. Lehmer: *On the values of certain Hurwitzian continued fractions.*

Euler was the discoverer of a number of remarkable regular continued fractions representing certain simple combinations of e and its rational powers as, for example, $(e+1)/(e-1) = [2, 6, 10, 14, \cdots]$ or $(e^2+1)/(e^2-1) = [1, 3, 5, 7, \cdots]$. In this paper we find the value of continued fraction $[b, a+b, 2a+b, \cdots]$, whose partial quotients form a general arithmetic progression, in terms of quotients of Bessel functions of imaginary argument. Only in case $2b/a$ is an odd integer is the continued fraction simply expressible in terms of rational powers of e. All continued fractions like Euler's $(e-1)^{-1} = [0, 1, 1, 2, 1, 1, 4, 1, 1, 6, \cdots]$, whose quotients are periodic except for a subsequence in arithmetic progression, are also evaluated as Bessel quotients, as well as certain continued fractions whose partial quotients are made up of two or more arithmetic progressions. Semiregular continued fractions whose numerators are all equal to -1 likewise may be evaluated in terms of Bessel quotients of real arguments. (Received March 8, 1941.)

214. C. D. Olds: *On some arithmetical identities.*

In this note the author considers certain arithmetical identities of the type: $\sum \cos [2(a+b)\pi/p]=\sum \cos [2(a-b)\pi/p]$, $b\neq a$, where a and b range successively over all the quadratic residues of the given prime number p. Some applications are indicated. (Received March 8, 1941.)

215. Max Zorn: *Transcendental p-adic numbers related to roots of unity.*

Elementary remarks about some sequences which converge p-adically. (Received March 10, 1941.)

Analysis

216. E. F. Beckenbach: *On functions having subharmonic logarithms.*

Two new characterizations of functions having subharmonic logarithms are given. One is expressed in function-theoretic terms, while the other is given by a generalized isoperimetric inequality. (Received March 10, 1941.)

217. Stefan Bergman: *The method of the minimum integral and the analytic continuation of functions.*

The measures of geometrical objects which occur in the theory of *CT*'s (conformal transformations) and *PT*'s (pseudo-conformal transformations) can be expressed as functions of minima $\lambda_{\mathfrak{B}}(t)$ of integrals $\int_{\mathfrak{B}} |h|^2 dx_1 dy_1 \cdots dx_n dy_n$, $z_k=x_k+iy_k$, where h runs through all functions analytic in $\mathfrak{B}$ and subjected to certain conditions at the point (t). If $\mathfrak{G}\subset\mathfrak{B}$ then $\lambda_{\mathfrak{G}}(t)\leqq\lambda_{\mathfrak{B}}(t)$, and since the equation $ds^2_{\mathfrak{B}}(z)=|dz|^2/\lambda_{\mathfrak{B}}(t)$ defines a metric invariant with respect to *CT*'s the lemma of Schwarz-Pick (see Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 16 (1937), p. 11) is obtained. On the other hand $\lambda_{\mathfrak{B}}(t)$ can be expressed with the aid of a system of functions orthogonal in $\mathfrak{B}$. Thus, in the case of Schlicht *CT*'s one obtains certain refinements of the lemma of Schwarz-Pick. The introduction of the concept of B-area enables the author to generalize this technique to *PT*'s. Finally, if one supposes that functions $w_k(z_1, z_2)$, $k=1, 2$, of the *PT* satisfy certain conditions on a surface bounding a segment $\mathfrak{a}$ of the boundary then w_1, w_2 are regular in $\mathfrak{a}$, and the *PT* (w_1, w_2) can be extended analytically through $\mathfrak{a}$ outside its original domain of definition. Applying then the method of the minimum integral for the extended domain, the author obtains results concerning distortion on the boundary for *PT*'s. (Received April 1, 1941.)

218. Lipman Bers: *On a generalized harmonic measure.*

Suppose D is a domain in R_n, B its (bounded) boundary, B' the set of the regular boundary points of D, $u(P)$ $(P\in D)$ a bounded harmonic function (b.h.f.). There exists a function $\mu_u(P, e)$ (generalized harmonic measure) which is a b.h.f. of P for every fixed Borelian set $e\subset B$, a completely additive set function for every fixed point $P\in D$ and satisfies the condition $\lim [\mu_u(P, e)-u(P)]=0$, $\lim \mu_u(P, B-e)=0$ $(P\to R\in B')$ for every open set $e\subset B$, $R\in e$. If $f(Q)$ is a continuous function, $v(P)=\int_B f(Q)d\mu_u(P, e_Q)$ is the solution of the following problem (a generalization of the Dirichlet problem): determine a b.h.f. v such that $\lim [v(P)-f(R)u(P)]=0$ $(P\to R\in B')$. If f is a bounded Baire function, v is a b.h.f. which is "representable by u." The bounded harmonic functions which possess a positive lower bound and which

are representable by a given function u form a "representation class" $C(u)$. Every function of $C(u)$ is representable by any other function of this class and only by these functions. In a given domain there exists either only one or continuously many different representation classes. (Received April 1, 1941.)

219. P. B. Burcham: *Some inclusion relations in the Hausdorff difference matrix.*

The matrix $(\Delta^m c_n)$, where $\{c_n\}$ is a regular Hausdorff moment sequence, has been defined as the difference matrix (Transactions of this Society, vol. 48 (1940), pp. 185–207). In this paper it is established that corresponding row sequences of the matrices $(\Delta^m a_n)$ and $(\Delta^m b_n)$ define equivalent methods of summation provided that the sequences $\{a_n\}$ and $\{b_n\}$ define the equivalent methods of summation, Cesàro summability $(C, \gamma-\alpha)$ and hypergeometric summability $(H, \alpha, 1, \gamma)$, $\Re(\alpha, \gamma, \gamma-\alpha)>0$. Additional results are obtained concerning the column and diagonal sequences of these two matrices. In the case that $\{a_n\}$ and $\{b_n\}$ define the Cesàro methods (C, α) and (C, β), $\Re(\alpha, \beta)>0$, it is found that all of the diagonal sequences of the associated difference matrices define equivalent methods of summation. (Received February 24, 1941.)

220. Richard Courant: *On a new type of variational problems, with physical demonstrations.*

The classical calculus of variations concerns joining extremals between two points, or spanned within a closed curve, or problems of a similar type. There are, however, problems of equilibrium in physics that suggest a more general type of variational problems, of which the following is an example: Given n points in the plane; find a simply connected graph of extremal arcs joining the n points and having stationary length in the variational metric. A famous classical problem by Steiner gives the simplest illustration. There is a connection between these problems and isoperimetric problems with inequalities as subsidiary conditions whereby in particular any ordinary variational problem appears as a limiting case of an isoperimetric problem. The position and the treatment of these general problems carries over to more dimensions; for example: Given in three dimensions a closed graph (say the system of edges of a cube); desired a system of minimal surfaces bounded by the graph and having common boundary lines to be determined, such that the total area becomes stationary. The problems, their solution, and their interdependence can be illustrated by simple soapfilm experiments. (Received March 31, 1941.)

221. E. L. Crow: *The expansion problem associated with an ordinary differential equation of first order in which the coefficient is a quadratic polynomial in the parameter.*

Consider the differential equation $dy(x, \lambda)/dx=p(x, \lambda)y(x, \lambda)$, where x is a real variable and λ is a complex parameter. The expansion problem associated with this equation and a two-point boundary condition has been discussed by R. E. Langer (Transactions of this Society, vol. 25 (1923), pp. 155–172) for the case in which $p(x, \lambda)$ has a finite number of simple poles as a function of λ. Convergence was established by expressing the series of characteristic solutions as a sum of residues at the characteristic values. However, the method fails when $p(x, \lambda)$ has a double pole, in particular when $p(x, \lambda)$ is of the form $a(x)\lambda^2+b(x)\lambda+c(x)$. The present paper ap-

proximates the circumstances of the latter problem by replacing one of the boundary points on the real axis by two neighboring points in the complex plane. The variable x is allowed to range over a simply-connected region including the three boundary points. An adaptation of a method developed by Langer (Transactions of this Society, vol. 46 (1939), pp. 151–190) for use in the complex domain is employed to establish the uniform convergence of simultaneous expansions of two arbitrary functions. (Received March 14, 1941.)

222. R. J. Duffin: *Möbius transforms and Fourier transforms.*

Let $g(x)$ be the Fourier cosine transform of $f(x)$. It is shown that formally $f(x)=(2\pi)^{1/2}\sum_{n=1}^{\infty}\mu_n/nx\sum_{m=1}^{\infty}g(2\pi m/nx)$. Here μ_n is the familiar Möbius symbol having the values 0, 1, -1. The application of suitable summability methods validates this inversion formula for a wide class of functions. It is sufficient to assume that $f(x)$ be absolutely integrable. Conditions may also be imposed on $g(x)$; for instance, the formula is valid if $g(x)$ is of bounded total variation. (Received March 13, 1941.)

223. Benjamin Epstein: *Asymptotic problems associated with a Laplace-Mellin integral equation.* Preliminary report.

In this paper the author studies the integral equation $\psi(s)=\int_A^{\infty}\sum_{n=0}^{\infty}(\lambda+n)^{-s}f(\lambda)d\lambda$, $A\geqq 0$, $\operatorname{Re} s>1$, $f(\lambda)$ L-integrable. In the first half of his paper he proves Abelian and Tauberian theorems for this equation relating the asymptotic properties of $\psi(s)$ as $s\to 1^+$ with those of $f(\lambda)$ as $\lambda\to\infty$. The technique utilized is the repeated application of Tauberian and Abelian theorems for the Laplace integral equation. The second half of the paper is devoted to a moment problem analogous to the Stieltjes moment problem over the infinite interval. For example, the following question is considered: Suppose s runs through the monotone non-decreasing set of real numbers σ_n, $n=1, 2, \cdots$, with limit point infinity; then are there conditions on the size of $\psi(\sigma_n)$ and the density of σ_n which assure uniqueness of solution? Analytic criteria for this problem are found. The chief tools utilized are theorems by Carleman and Junnila for functions analytic in a half-plane and a theorem of Boas (Transactions of this Society, vol. 46, pp. 142–150). (Received March 11, 1941.)

224. H. H. Goldstine: *The parametric problem of the calculus of variations in general analysis.*

The well known problem of the calculus of variations is here extended so that the integrand is defined on a region of the spaces of points (t, x_p, r_p), where p ranges over an arbitrary class P. Analogues of the usual necessary conditions, and at least some of the sufficiency conditions, are obtained. The principal point of departure from the results for the problem in finitely many dimensions is in the discussion and character of extremals. (Received March 11, 1941.)

225. W. J. Harrington and J. B. Rosser: *A study of certain functions auxiliary to Brun's method in number theory.*

In 1937 J. B. Rosser made a preliminary report on *An improvement of Brun's method in number theory* (this Bulletin, abstract 43-3-142). Certain upper and lower bounds stated in that report were obtained by approximating certain functions by very extensive computations. In the present paper these functions are studied and much superior methods of evaluation are devised. The functions consist of infinite

series, the terms of which are multiple integrals. The principal aim of the paper is to establish certain identities which express the sums of the series in closed form. Preliminary to proving these identities, one has to prove convergence, continuity, and existence of derivatives. These matters take up the bulk of the paper. In the report mentioned above, the results stated were relevant only to a particular application of Brun's method. The functions studied in this paper are taken to be sufficiently general so that the evaluations obtained can be used in the entire range of application of Brun's method. This paper prepares the way for the publication of the paper of which the preliminary report was made in 1937. (Received March 26, 1941.)

226. J. F. Heyda: *A uniqueness property of general monogenic functions.*

Let $f(\alpha)$ be the class of functions defined by $f(\alpha)=\iint q(x, y)(z-\alpha)^{-1}dxdy$ (α in E) integrated over the complement $C(E)$ of a closed set E in a bounded domain K of the α-plane. The function $q(x, y)$ is real and continuous in $\overline{K}$ and vanishes at points of E; moreover $z=x+iy$, where (x, y) is a point in $C(E)$. The form of the functions $f(\alpha)$ is suggested by the non-analytic part of the derivative of $F(\alpha)$, where $F(\alpha)$ is general monogenic in E (as defined by W. J. Trjitzinsky, Annales de l'École Normal Supérieure, vol. 55, pp. 119–191) and is represented there by $F(\alpha)=h(\alpha)+\iint_{C(E)}\log(z-\alpha)q(x, y)dxdy$. Suppose E contains a closed interval I and Γ is a closed linear set of points in I having positive measure. The following uniqueness problem is considered: what must be the "rarefication" of the set $C(E)$, or, alternately, how fast must $|q(x, y)|\to 0$ as $\rho\to 0$ (ρ is the distance of (x, y) from the frontier of $C(E)$) in order that knowledge of the functional values of $f(\alpha)$ on Γ will determine uniquely the values of $f(\alpha)$ on Γ', where $I\supset\Gamma'\supset\Gamma$ and measure $\Gamma'>$measure Γ. (Received February 28, 1941.)

227. Dunham Jackson: *Generalization of a theorem of Korous.*

An elementary treatment of the convergence of series of orthogonal polynomials is greatly facilitated if the polynomials of the orthonormal set are known to be uniformly bounded on the domain of orthogonality, or on a part of it where convergence is to be proved. A demonstration due to J. Korous (see G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, p. 157) shows in a few lines that the orthonormal polynomials corresponding to a weight function $\rho\sigma$ on an interval are thus bounded, if the polynomials for weight ρ have the desired property, and if the factor σ satisfies a Lipschitz condition and has a positive lower bound on the entire domain of orthogonality. The purpose of this note is to show that the argument of Korous can be extended so as to apply under fairly general conditions to orthogonal polynomials in two real variables on an algebraic curve, and in particular to orthogonal trigonometric sums, which can be regarded as orthogonal polynomials on a circle. (Received February 6, 1941.)

228. H. Kober: *On a theorem of Schur and on fractional integrals of purely imaginary order.*

Schur's theorem ideals with linear transformations on L_2 to L_2 involving a kernel $K(x, y)$, homogeneous of degree -1, such that $K(x, 1)x^{-1/2}\in L_1(0, \infty)$. It is shown that two types of fractional integrals of order α belong to this class of transformations for $\Re(\alpha)>0$ and tend to bounded limits in $L_2(0, \infty)$ as $\Re(\alpha)\to 0$. The inversion and group properties of such fractional integrals of purely imaginary order are studied.

The results admit of partial extensions to L_p. Characteristic values and functions are also determined. (Received February 13, 1941.)

229. R. E. Lane: *The first three rational operations.*

This paper establishes the foundation for an approach to the number system in a manner designed to postpone the need for infinite processes such as Dedekind cuts. In contrast to the Cantor and Dedekind approaches, it considers numbers as being generated from the unit-elements of a sequence of rational operations analogous to addition and multiplication. Postulates are laid down for such a sequence of operations, and a number of theorems are derived for the first three rational operations: intersection, union, and addition. (Received March 4, 1941.)

230. Lincoln La Paz: *Double integral variation problems with prescribed transversality coefficients.*

H. A. Simmons has recently published an interesting derivation of the transversality relationship for the variable limit problem of the calculus of variations for n-tuple integrals (Transactions of this Society, vol. 36 (1934), pp. 29–43). It is the purpose of the present note to formulate and solve an inverse problem suggested by this transversality relationship. Attention is restricted to the double integral case but the argument made and the conclusions drawn are easily extended to n-tuple integrals. Transversality coefficients are defined for a regular double integral problem. Necessary and sufficient conditions for a pair of functions to be transversality coefficients of such a problem are obtained and the integrand function of the most general variation problem associated with a pair of transversality coefficients is determined. (Received March 13, 1941.)

231. Hans Lewy: *Fractional potentials.*

In the theory of the logarithmic potential $\log r$, Laplace's formula gives the distribution of mass in terms of the potential. The author derives an analogous explicit formula for potentials $r^{-\beta}$ with β constant and $0<\beta<2$. Under certain rather general conditions, for a given function U, a mass distribution is determined whose potential is U. Similar formulae hold in spaces of arbitrary dimension. They can be utilized in the construction of the conductor potential by reducing this problem to that of an integral equation of second kind. The restriction of β to negative numbers may be dropped. (Received February 25, 1941.)

232. A. T. Lonseth: *Dirichlet's principle for certain nonlinear elliptic equations.*

Courant's "Dirichlet principle" method of solving the Dirichlet problem for self-adjoint linear partial differential equations of elliptic type (*Methoden der Mathematischen Physik*, vol. 2, Berlin, 1937) is extended to nonlinear equations of type $\Delta u = 1/2\ \partial P(x, y; u)/\partial u$, where Δ is the Laplacian differential operator and $P(x, y; u)$ is non-negative, analytic in its three arguments, and convex in u. The (unique) solution automatically minimizes the integral whose first variation it nullifies. An alternative proof of the minimizing property is obtained. Previous existence proofs for this type of nonlinear elliptic equation are due to Bieberbach (method of successive approximations) and to Lichtenstein (method of Ritz). Analyticity of $P(x, y; u)$ may be replaced by less stringent conditions, and Δ by a more general self-adjoint linear elliptic operator. (Received March 12, 1941.)

233. L. H. Loomis: *A converse to the Fatou theorem.*

Let $f(z)$ be analytic and bounded in $|z|<1$ and let $F(z)$ be an indefinite integral of $f(z)$. Then $F(z)$ is continuous on the closed circle $|z|\leq 1$ and defines a boundary function $F(t)$, $t=e^{i\theta}$. The proof of the Fatou theorem consists of showing that $f(z)$ has a radial limit equal to the derivative $F'(t)$ wherever $F'(t)$ exists. In this note the author proves conversely that wherever the radial limit $f(e^{i\theta})=\lim_{r\to 1} f(re^{i\theta})$ exists the derivative $F'(t)=F'(e^{i\theta})$ exists and has the same value. Thus the boundary function $f(t)$ is identically equal to the derivative of its indefinite integral, in the sense that the two functions exist at precisely the same points and have there the same values. An example is constructed to show that the hypothesis of boundedness cannot be materially weakened. (Received April 1, 1941.)

234. L. H. Loomis: *A simple proof of the Fatou theorem.*

The Fatou theorem in its simplest formulation can be stated as follows: If $f(x)$ is analytic and bounded in the unit circle $|z|<1$ then the radial limit $\lim_{r\to 1} f(re^{i\theta})$ exists for almost all θ. The standard proof of this theorem is based on the fact that a function of a real variable having a Lipschitz constant has a derivative almost everywhere. The manipulations involved center around the Poisson integral formula. The present proof assumes the same basis but attains essential manipulatory simplification by using the Cauchy integral formula instead of the Poisson integral and substituting the half-plane for the circle. (Received April 1, 1941.)

235. E. R. Lorch: *The spectrum of linear transformations.*

Let T be a bounded linear transformation in a complex Banach space, and let C be a simple closed rectifiable curve lying entirely within the resolvent set R of T. Then the integral $(1/2\pi i)\int_C d\zeta/(\zeta I-T)$ exists and represents a projection P reducing T. $P=0$ if and only if every point interior to C lies in R. $P=I$ if and only if every point exterior to C lies in R. If C' represents another curve in R and P' its associated projection, then $PP'=P$ if C lies in the interior of C'; $PP'=0$ if C and C' lie exterior to each other. This leads to the construction of a homomorphism between an algebra of sets on the one hand and an algebra of projections reducing T on the other—this mapping giving the spatial manifold counterparts of a fundamental class of spectral sets. (Received February 13, 1941.)

236. K. L. Nielsen: *Concerning boundary value problems for linear differential equations when the boundary conditions are given by Stieltjes integrals.*

This paper deals with the boundary values problem formulated by $L(x, \lambda; y(x, \lambda))=f(x)$ and the boundary conditions $M_i(y)\equiv\sum_{k=1}^{n}\int_c^d y^{(k-1)}(t, \lambda)d\alpha_{i,k}(t)=0, (i=1, \cdots, n)$, where $L(x, \lambda; y(x, \lambda))$ is the differential polynomial $\sum_{k=0}^{n}\lambda^{H(n-k)}a_{n-k}(x, \lambda)y^{[k]}$, the operators M_i are linearly independent, the involved integrals are in the sense of Stieltjes and the $\alpha_{i,k}(x)$ are functions of bounded variation. Tamarkin (Mathematische Zeitschrift, vol. 7 (1927), pp. 1–54) presented developments under the assumption that the roots of the characteristic equation are distinct and that $H=1$. Trjitzinsky (Acta Mathematica, vol. 67 (1936), pp. 1–50) has obtained an asymptotic representation of the solution of the differential equation for x in the real interval (c, d) and the parameter λ in certain regions extending to infinity for the case where the roots of the characteristic equation are not distinct and H is allowed to exceed

unity. The author takes Trjitzinsky's results and obtains the restrictions which are necessary on the $\alpha_{i,k}(x)$ and states a theorem determining the values of λ for which the nonhomogeneous boundary value problem is possible when (c, d) is the interval for which the Trjitzinsky existence theorem holds. (Received March 31, 1941.)

237. C. D. Olds: Expansion of an arbitrary function in terms of certain polynomials.

Let $\{P_i(z)\}$ be a set of polynomials which satisfies an equation of the type: $Q(z)/n! = \sum_{i=0}^{n} \alpha_i P_i(z)$. (When $\{P_i(z)\}$ are the Bernoulli polynomials, for example, $Q(z) = z^n$.) This property is used to show that under certain conditions an arbitrary function $f(z)$, regular in the unit circle, can be expanded in the form $f(z) = \sum_{i=0}^{\infty} \beta_i P_i(z)$. Moreover, for certain of these polynomials the expansion is shown to be valid by summing the series directly. In the case of the Bernoulli polynomials use is made of certain results concerning their maximum values due to D. H. Lehmer (American Mathematical Monthly, vol. 47 (1940), pp. 533–538). (Received March 8, 1941.)

238. G. H. Peebles: *On the behavior of series of polynomials orthogonal with respect to a weight function of changing sign.*

Most of the properties of polynomials orthogonal with respect to a non-negative weight function, which are needed to show convergence of the formal expansion of a function, are lost, if the weight function changes sign in the interval of orthogonality. The partial sums corresponding to a weight function $\rho(x)$ which changes sign a finite number of times are related very simply, however, to the partial sums corresponding to $\pi(x)\rho(x)$, where $\pi(x)$ is a polynomial or reciprocal of a polynomial such that $\pi(x)\rho(x)$ is non-negative on the interval of orthogonality. This relation is used to study the behavior of the partial sums of the expansions of functions in terms of polynomials orthogonal with respect to a weight function of changing sign. (Received March 14, 1941.)

239. Tibor Radó: *On convergence in length and convergence in area.*

The purpose of the paper is to complete and to extend various results in the literature on convergence in length and convergence in area. The specific tool is the following inequality. On a bounded measurable set E of a euclidean space, let there be given two three-dimensional vector-functions x_1, x_2. Then $[\int \|x_1 \times x_2\|]^2 \leq [\int \|x_1\| \cdot \|x_2\|]^2 - [\int x_1 \cdot x_2]^2$, where the integrals are taken over E, provided only that the integrals involved exist in the Lebesgue sense. This inequality, which is an immediate consequence of the identity $(x_1 \times x_2)^2 = x_1^2 x_2^2 - (x_1 \cdot x_2)^2$, is a generalization of the inequality (7) in Adams and Lewy, *On convergence in length*, Duke Mathematical Journal, vol. 1 (1935), pp. 19–26. (Received March 28, 1941.)

240. W. C. Randels: *On Bessel's inequality in abstract spaces.*

Bochner (Fundamenta Mathematicae, vol. 20, p. 262) has defined Fourier series for functions whose values are in abstract spaces, and has pointed out that Bessel's inequality need not hold. In this paper examples of spaces in which the inequality does hold are given and Bochner's counterexample is extended. (Received March 3, 1941.)

241. C. E. Rickart: *Integration in a convex linear topological space.*

A theory of integration is developed for multi-valued functions $F(\sigma)$ defined over a σ-field $\mathfrak{M}$ of subsets of an abstract set M and with values in a convex linear topo-

logical space $\mathfrak{X}$. The value of the integral is a closed set in $\mathfrak{X}$. The fundamental properties obtained by Kolmogoroff (Mathematische Annalen, vol. 103 (1930), pp. 654–696) for $\mathfrak{X}$ the space of real numbers are shown to carry over to the more general situation. The method of definition of the integral is an extension of that used by R. S. Phillips (Transactions of this Society, vol. 47 (1940), pp. 114–145). A notion of convergence for subsets of $\mathfrak{X}$ similar to the Hausdorff convergence for subsets of a metric space plays a central role throughout. If the integral is restricted to be single-valued, then a very general convergence theorem involving a notion of approximate convergence is obtained. The integral contains an integral of G. B. Price (Transactions of this Society, vol. 47, pp. 1–50) as well as the Kolmogoroff and Phillips integrals. (Received March 14, 1941.)

242. Raphaël Salem: *On some properties of symmetrical perfect sets.*

I. Let P be a symmetrical perfect set of measure zero constructed in $(0, 2\pi)$ by the dyadic process; E_p the set of 2^p equal intervals obtained after p operations; F_p the continuous non-decreasing function $(F_p(0)=0,\ F_p(2\pi)=1)$ equal to $k/2^p$ in the kth interval contiguous to E_p, and linear in each interval of E_p; F the limit of F_p. If c_n is the Fourier-Stieltjes coefficient of dF with respect to e^{nix} and $E(p)$ the measure of E_p, the series $\sum E^2[a \log n] \cdot |c_n|^2$ converges for $a > 3/2 \log 2$, its sum depending only on a. Thence follow some properties of sets of absolute convergence for trigonometrical series. II. The following theorem is proved: if N is any set of absolute convergence for trigonometrical series, the sum of N and of any finite set is also a set of absolute convergence. (Received March 6, 1941.)

243. Raphaël Salem: *On trigonometrical series whose coefficients do not tend to zero.*

If $\sum \rho_n \cos(nx-\alpha_n)$, with $\rho_n \geqq 0$, $\limsup \rho_n > \alpha > 0$, converges in a set E, and if the point $x=0$ belongs to E, then there is a sequence of integers $\{n_k\}$ such that $\sin n_k x$ tends to zero for every x belonging to E. If E is perfect and if F is any non-decreasing function constant in every interval contiguous to E but increasing from one interval to another, the Fourier-Stieltjes cosine-coefficients of rank $2n_k$ of dF tend to the greatest possible limit $(F(2\pi)-F(0))/\pi$ as $k \to \infty$. If E is a perfect set such that this condition is satisfied, then there exists a trigonometrical series whose coefficients do not tend to zero and which converges "almost everywhere" in the set E. (Received March 6, 1941.)

244. Henry Scheffé: *Linear differential equations with two-term recurrence formulas.*

Linear differential equations with two-term recurrence formulas (defined below) occur very often in applied mathematics. The equation $\sum_{j=0}^{n} p_j(x) y^{(j)} = 0$ is assumed to have coefficients analytic at some $x=x_0$. Following Frobenius, a formal substitution $y=\sum_{\nu=0}^{\infty} c_\nu \xi^{\nu+r}$ is made where $\xi \equiv x-x_0$ if $x_0 \neq \infty$, $\xi \equiv 1/x$ if $x_0 = \infty$. The formula obtained by equating to zero the coefficient of $\xi^{\mu+r}$ for integral values of μ is called the recurrence formula relative to the point x_0. For the existence of a two-term recurrence formula, that is, one of the type $f(\nu, r)c_\nu + g(\nu, r)c_{\nu-h} = 0$, for $\nu \geqq h$, a positive integer, necessary and sufficient conditions on the coefficients $p_j(x)$ are obtained for some neighborhood of x_0. It is then further assumed that the $p_j(x)$ are given elsewhere by their analytic continuations. Transformations $z=a\xi^b$, $w=\xi^c y$ are exhibited which carry the

differential equation into a generalized hypergeometric equation with indices $p \leqq q+1$. Interesting corollaries follow. (Received March 12, 1941.)

245. I. J. Schoenberg: *On absolutely convex functions.*

1. A function $f(x)$ is said to be *completely monotone* for $x>0$ if (1) $(-1)^n f^{(n)}(x) \geqq 0$ $(x>0;\ n=0, 1, 2, \cdots)$. The following theorem shows that these requirements are redundant. Theorem: $f(x)$ is completely monotone for $x>0$ if it satisfies, for $x>0$, the following conditions: (2) $f(x) \geqq 0$, $f'(x) \leqq 0$, $f''(x) \geqq 0$, and (3) $(-1)^n f^{(n)}(x) \geqq 0$ for infinitely many values of n (no matter how distantly spaced). Moreover, the third condition (2) may be replaced by the following weaker one: $\liminf_{x \to +\infty} f'(x) > -\infty$. 2. A function $f(x)$ is said to be *absolutely convex* for $-1<x<1$ if it satisfies in that range the inequalities $f^{(2n)}(x) \geqq 0$ $(n=0, 1, 2, \cdots)$. S. Bernstein has shown that $f(x)$ is analytic and regular in the domain bounded by the circle $|x| = 1/4$ and the four tangents to it through the points $x=1$, $x=-1$. It is shown here that $f(x)$ is analytic within the unit circle $|x|<1$, which is the best result possible. (Received February 7, 1941.)

246. H. M. Schwartz: *Sequences of Darboux-Stieltjes integrals.*

The results obtained for sequences of Stieltjes integrals (abstract 47-1-43) are extended to sequences of upper and lower Darboux-Stieltjes integrals taken in the sense of Lebesgue (*Leçons sur l'Intégration*, 2d edition). (Received March 1, 1941.)

247. W. R. Scott: *A lemma on the Weierstrass E-function.* Preliminary report.

Let there be given an integrand $f(x^1, x^2, x^3, X^1, X^2, X^3)$ which we abbreviate $f(x, X)$. We assume that f satisfies the usual differentiability and homogeneity conditions. Let there be given further a set of values x_0, X_0 such that (1) $f(x_0, X_0)>0$, (2) $f(x_0, X) \geqq 0$ and $E(x_0, X_0, X) \geqq 0$ for every vector X, where E is the Weierstrass E-function. Let Y_0 denote the vector with components $\partial f/\partial X^i$, $i=1, 2, 3$, taken at (x_0, X_0). Then there exists a $\delta = \delta(x_0, X_0)>0$ such that $f(x_0, X) \geqq \delta$ for every unit vector X such that the scalar product XY_0 is positive. This lemma is then applied to answer various questions concerning the lower semi-continuity of simple and double integrals suggested by the following papers; E. J. McShane, *Integrals over surfaces in parametric form*, Annals of Mathematics, (2), vol. 34 (1933); *Semi-continuity of integrals . . .*, Duke Mathematical Journal, vol. 2 (1936); T. Radó, *On the semi-continuity of double integrals in parametric form*, to appear in Transactions of this Society. (Received March 12, 1941.)

248. W. T. Scott and H. S. Wall: *Gronwall summability.*

It is shown that a necessary and sufficient condition for a Hausdorff mean $[H, c_n]$ to be a Gronwall mean (f, g) (T. H. Gronwall, *Summation of series and conformal mapping*, Annals of Mathematics, (2), vol. 33 (1932), pp. 101–117) is that $[H, c_n]$ be the product of (E, β) by (C, α), that is, $c_n = \beta^n / C_{n+\alpha, n}$, $(n=0, 1, 2, \cdots)$, where $0<\beta \leqq 1$, $\alpha>0$. The (f, g)-mean which is identical with this $[H, c_n]$-mean has $f(w) = \beta w/[1-(1-\beta)w]$, $g(w) = (1-w)^{-\alpha-1}$. It is also shown that a method of summation considered by W. A. Mersman (this Bulletin, vol. 44 (1938), pp. 667–673) is identical with (f, g) with $f(w) = [1-(1-w)^{1/2}]/[1+(1-w)^{1/2}]$, $g(w) = (1-w)^{-1}$. (Received February 25, 1941.)

249. J. A. Shohat: *On the best polynomial approximation for functions possessing derivatives.*

By virtue of the characteristic property of the polynomial of best approximation, of degree less than or equal to n, to $f(x)$ on a given interval, it may be also considered as a Lagrangean interpolation polynomial for $f(x)$. In this way are obtained very simply general theorems on best approximation in case $f^{(n+1)}(x)$ exists, also on the distribution of the points of deviation. (Received February 3, 1941.)

250. L. H. Swinford: *On Abel's equation.*

The substitution $y=\alpha+\beta z^{1/2}/\gamma+z^{1/2}$ allows one to replace Abel's equation $y'+g(x)y^2+f(x)y^3=0$ by a system of two first order equations, by which means new cases of integrability are found. (Received March 7, 1941.)

251. Otto Szász: *On convergence and summability of trigonometric series.*

A new type of necessary or sufficient condition has been given recently for the convergence of a Fourier series at a point. This combines a continuity property of the function with an order condition on the coefficients, and is generalized in this paper. Moreover, instead of a Fourier series, more general trigonometric series are considered, associating for example the termwise integrated series with a function. The results are closely connected with two summability methods introduced by Riemann and Lebesgue. (Received March 5, 1941.)

252. Alexander Weinstein: *On the decomposition of a Hilbert space by its harmonic subspace.*

It is shown that a lemma playing a central part in a recent paper of H. Weyl on the method of orthogonal projection in potential theory (Duke Mathematical Journal, vol. 7 (1940), pp. 411–444, Lemma 2) is an almost immediate consequence of a problem of the unified theory of eigenvalues of plates and membranes considered some years ago in a joint paper of N. Aronszajn and the author (Comptes Rendus de l'Académie des Sciences, Paris, vol. 204 (1937), p. 96). The proof based on these results does not require any special construction or computation. (Received February 7, 1941.)

Applied Mathematics

253. G. E. Hay: *The finite displacement of thin rods.*

The finite displacement of thin rods has been considered by G. Kirchhoff, who introduced approximation based on the thinness of the rod in a rather unsatisfactory manner. In the present paper the method of the tensor calculus is employed, and there is introduced a systematic method of approximation which involves the expansion of the fundamental equations as power series in a dimensionless parameter ϵ and permits a theoretical solution of the problem to any desired degree of accuracy. Finally, application of the theory is made to the problem of "straightening" certain thin rods by means of systems of forces applied to the ends. (Received March 13, 1941.)

254. A. E. Heins: *On the transformation theory of the solution of partial differential equations.* Preliminary report.

The regularity condition of the Fourier transform of a function defined over a

finite domain, and zero outside of this domain, plays an important role in pure mathematics. This condition also plays an important role in the transformation theory of the solution of linear partial differential equations of mathematical physics. By observing this regularity condition, boundary conditions which do not enter into a particular problem may be eliminated. With the aid of the finite Fourier transform the author obtains solutions of the two-dimensional wave equation and the equation of heat conduction under prescribed initial and boundary conditions and the two-dimensional Laplace equation under prescribed boundary conditions. The boundaries are taken to be either circles or rectangles. The treatment of these problems with other contours is now in progress. (Received March 10, 1941.)

255. A. S. Householder: *A theory of steady-state activity in nerve-fiber networks.* II. *The simple circuit.*

Let the neuron N_i have the origin s_i and terminus s_{i+1}, subscripts being reduced modulo n. These neurons constitute a simple circuit. A constant stimulus y_i applied to N_i at s_i is supposed to cause the production at s_{i+1} of a stimulus z_{i+1} according to the equation $z_{i+1}=\alpha_i(y_i-h_i)$, where $h_i>0$, $\alpha_i=0$ if $y_i\leqq h_i$, $\alpha_i=a_i>0$ or <0 if $y_i>h_i$. If S_i is a constant stimulus applied from outside the circuit at s_i, then $y_i=S_i+z_i$. Then (1) the S_i do not necessarily suffice to determine the α_i, and hence, the y_i, unless $a_1a_2\cdots a_n<1$. If $(\alpha_1, \cdots, \alpha_n)$ and $(\alpha_1', \cdots, \alpha_n')$ are distinct and both consistent with some $(S_1, \cdots, S_n)$, and if some $\alpha_i=0$ and also some $\alpha_j'=0$, then (2) $\alpha_i=0$ implies $\alpha_i'\neq 0$; (3) $\alpha_i=\alpha_j'=0$ implies $a_ia_{i+1}\cdots a_{j-1}<0$; (4) $\alpha_i=\alpha_j=0$ implies $a_ia_{i+1}\cdots a_{j-1}>0$. (Received March 8, 1941.)

256. William Mersman: *Heat conduction in an infinite composite solid.*

The problem of one-dimensional heat conduction in a doubly infinite composite solid, with given initial temperature distribution, is solved uniquely. The classical solution for a semi-infinite solid with surface temperature known is used to obtain an integral equation, which is solved formally by means of the Laplace transformation. The solution is then established and proved unique by classical methods. (Received March 8, 1941.)

257. I. S. Sokolnikoff: *Deflection of non-isotropic elastic plates.* Preliminary report.

The general solution of the equation for the deflection w of a non-isotropic plate with one plane of elastic symmetry, $\alpha_{11}\partial^4w/\partial x^4+2\alpha_{12}\partial^4w/\partial x^3\partial y+2\alpha_{13}\partial^4w/\partial x^2\partial y^2+2\alpha_{31}\partial^4w/\partial y^3\partial x+\alpha_{22}\partial^4w/\partial y^4=p(x, y)$, can be written in terms of two analytic functions $F_1(z_1)$ and $F_2(z_2)$ of the independent complex variables $z_1=x+k_1y$ and $z_2=x+k_2y$, where the complex numbers k_i depend on the elastic constants α_{ij}. The determination of the functions F_i can be reduced to the search for certain analytic functions of a complex variable ζ related to z_1 and z_2 via a pair of mapping functions $z_i=\omega_i(\zeta)$. The development of the theory is analogous to the treatment of certain two-dimensional problems in isotropic theory initiated by N. Muskhelišvili (Mathematische Annalen, vol. 107 (1932), pp. 282–312). Some applications of the theory to a clamped elliptic plate are made. The formulas specialize to the known results of the theory of isotropic and orthotropic plates. (Received March 8, 1941.)

258. R. M. Sutton: *An instrument for drawing confocal conics.*

A simple device is demonstrated for drawing families of orthogonal ellipses and hyperbolae. On a spring roller is wound a fishline in two strands so that as the line is pulled out both strands elongate at the same rate and are held under tension by the roller. Confocal hyperbolae are described by holding chalk against the cord without slippage; confocal ellipses are drawn in the usual manner by slipping the chalk within a loop of fixed length. The eccentricity of either kind of curve is changed with ease, and the distance between focal points can be altered quickly. The instrument offers convenience, flexibility, and rapidity of operation. (Received March 28, 1941.)

259. C. J. Thorne and J. V. Atanasoff: *The application of a general functional approximation method to thin plate problems.*

The approximate solutions to two problems of thin plate theory are obtained by a general functional method. This method includes, indeed unifies into a connected whole, many of the standard approximation methods and many more. In particular it includes the methods of Trefftz, Boussinesq or "least square," and Rayleigh-Ritz. The first problem represents that of a centrally loaded, square, clamped plate. The solution to this problem illustrates convergence to boundary values as well as the differential equation. The solution agrees accurately with the results obtained by others. The second problem is that of an infinite, corner loaded plate supported by an elastic foundation. A solution containing seventeen constants is obtained. (Received March 31, 1941.)

260. Alexander Weinstein: *On the vibrations of a clamped plate under tension.*

This problem, of importance for acoustic reception, has been solved in terms of Bessel's functions in the elementary case of a circular plate by W. G. Bickley, Philosophical Magazine, vol. 15 (1933), pp. 776–797. It will be shown here, *for a plate of any shape*, that a convergent sequence of *lower bounds* for the frequencies can be computed by using the solutions of the membrane problem for the same domain. In the case of a rectangular plate the resulting equations involve only trigonometric and hyperbolic functions and can be more easily solved numerically than the equations for the circular plate. *Upper bounds* for the frequencies can be computed by the Rayleigh-Ritz method. However, lower bounds are more important in the present case. (Received April 1, 1941.)

GEOMETRY

261. Reinhold Baer: *Homogeneity of projective planes.*

It is the main object of this note to show that the theorems of Desargues and Pappus are valid in a projective plane if and only if there exist two one-parameter families of dualities of this plane which meet certain requirements (like existence of axis and center). (Received March 12, 1941.)

262. Y. K. Bal: *Study on graphs.* Preliminary report.

This is an abstract study of generalized polygons and their properties invariant under the independent rigid motion of their vertices. A relation, called "next," is introduced, in general by: I. If $A \leftrightarrow B$ then $B \leftrightarrow A$, II. $A \leftrightarrow A$ cannot hold; in particular by: II′. If $A \leftrightarrow B \leftrightarrow C$ then not $A \leftrightarrow C$. I and II′ give II. (a) S_n is not null. (b) If

$A \in S$ there are n and only n elements next to it. (c) If A, $B \in S_n$ there are $(A_1, \cdots, A_n) \in S_n$ for which $A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_r \leftrightarrow B$. n is finite and r is finite or zero. $|S_n|$ is the number of elements in S_n. Definition of $S_{n,p}$: For distinct elements $A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_q$ and not $A_i \leftrightarrow A_j$ when $1 < |i-j| < p-1$. If $A_i \leftrightarrow A_p$ whenever $|i-j| = p-1$, $S_{n,p}$ is shown by $S'_{n,p}$. It is proved: $|S'_{n,4}| = 2n$ and $S'_{n,5}$ cannot exist. For distinct elements $A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_r \leftrightarrow A_1$ is a cycle C_n. A cycle is either S_1 or S_2. If $|C_n| = |S_n|$ it is a boundary cycle. The order of S_n is the number of distinct b-cycles not differing by circular permutation. The order of $S'_{n,4}$ is shown equal to $(n!)^2/n$. For A_1 and B_1, $A_1 \leftrightarrow \cdots \leftrightarrow A_{m_1} \leftrightarrow B_1$, then the smallest m_i is the distance (A_1, B_1) with $(A, A) = 0$. It is shown that $|S_n|$ is finite and by further study S_n's are classified. (Received March 31, 1941.)

263. Nathaniel Coburn: *Frenet formulas for curves in unitary space.*

The object of this paper is to develop a set of Frenet formulas for any curve X_1 whose equations are functions of a real parameter (t) and which lies imbedded in a unitary space of n dimensions K_n. The derivation proceeds analogously to that of the Frenet formulas for a curve V_1 which lies imbedded in a Riemannian space of n dimensions, V_n. However, in the Riemannian case, the curve possesses $(n-1)$ independent curvatures; in the unitary space, the curve possesses $(2n-1)$ independent curvatures. Furthermore, in Riemannian space, the curvature matrix of V_1 in V_n is skew-symmetric. Similarly, the curvature matrix of X_1 in K_n possesses Hermitian symmetry. Finally, corresponding to the similar theorem in Riemannian space, it is shown that "a complex curve X_1 in a unitary space K_n is uniquely determined when: (1) an initial point; (2) an initial orientation of the n-uple of normals; and (3) the $(2n-1)$ independent curvatures are given." The paper concludes with a study of geodesic X_1 in a unitary K_n with semi-symmetric connection. It is shown that these geodesics are characterized by the first Frenet curvatures and the orientation of the n-uple of normals. (Received February 11, 1941.)

264. V. G. Grove: *The transformation T of congruences.*

In a projective space of three dimensions two congruences are said to be related by a transformation T if their lines are in one-to-one correspondence, their developable surfaces correspond, and each congruence possesses transversal surfaces whose tangent planes at the points of intersection with a line of that congruence pass through the corresponding line of the other congruence. The transformation T is of two types. In the first, called the asymptotic type, the curves on the transversal surfaces of the congruences corresponding to developables of these congruences are asymptotic curves. If one of the congruences of the pair is a W-congruence, the other is also. In the second or conjugate type the curves on the transversal surfaces corresponding to the developables form conjugate nets in relation F. If these nets are in the relation of a transformation K of Koenigs, they are in the relation of Eisenhart. Associated with the conjugate case there exists a one-parameter family, or pencil, of congruences such that each focal point of a line of a variable member of the pencil lies on a line. If one congruence of this pencil is a W-congruence, all congruences of the pencil are W. (Received March 8, 1941.)

265. Edward Kasner and John De Cicco: *Geometry of dual-velocity systems.*

A dual-velocity system consists of the ∞^2 curves such that the osculating circles of the ∞^1 curves tangent to any line l, constructed at the elements of l, are tangent to

another line L. Any such set may be considered to be analogous to the velocity systems as developed by Kasner. Any line transformation, not preserving all parallel pencils of lines, converts exactly one dual-velocity system into a dual-velocity system. The group preserving all dual-velocity systems is $X=\phi(x)$, $Y=y\psi(x)+\chi(x)$. This is the contact group leaving invariant the set of all dual-isothermal families. Examples of dual-velocity systems are equitangential, dual-natural, Δ, and dual-Γ families. Characterizations of these are obtained by the correspondence between the lines l and L mentioned above. Any dual-velocity system contains exactly ∞^2, ∞^1, one, or zero dual-isothermal families. Finally, the invariant theory of dual-velocity systems under both the dual-isothermal and equilong groups is developed. A dual-analogue of natural family has been discussed in an earlier paper. (Received February 11, 1941.)

266. L. J. Savage: *Distance spaces.*

By a distance space is meant a set M of elements $p, q, \cdots$ over which is defined a real valued function $D(p, q)$ such that $D(p, q)=D(q, p)$, and $D(p, p)=0$. $D(p, q)$ may be thought of as the square of the distance from p to q. This concept is some generalization of metric space. A vector space V over which a scalar product $x\cdot y$ is defined can be considered as a distance space by setting $D(x, y)=(x-y)\cdot(x-y)$. For every distance space M there is a "smallest" scalar product space $V(M)$ in which M is imbeddable. An interesting class of distance spaces is that of differentiable manifolds over which a $D(p, q)$ is so defined as to be suitably differentiable when considered as a function of the coordinates. Because of the remark about scalar product spaces these differentiable distance manifolds can be handled much like differentiable submanifolds of euclidean space. In particular the concepts of regularity, tangent-flat, and first and second fundamental form, can be extended to them. Finally there are theorems connecting the possibility of imbedding such manifolds into euclidean and pseudo-euclidean spaces with certain restrictions on the second fundamental form. (Received March 13, 1941.)

267. R. K. Wakerling: *On the rational loci of ∞^1 $(\rho-1)$-spaces in r-space.*

The representation upon a ρ-space of the hypersurface W_ρ^n in S_r, which is the rational locus of ∞^1 $(\rho-1)$-spaces, is investigated in this paper. The hyperplane sections of W_ρ^n are represented in S_ρ by a system of hypersurfaces $V_{\rho-1}^\nu$ passing through a given $(\rho-2)$-space $\nu-1$ times, and having in common σ simple points. Some of the properties of W_ρ^n are discussed together with those of its projection upon a $(\rho+1)$-space. A special case of the transformation between two r-spaces is given, and the paper is concluded with a brief note on rational ruled surfaces of order $r-1$ in S_r. (Received March 8, 1941.)

Statistics and Probability

268. A. H. Copeland: *If.*

If one attempts to apply Boolean algebra to the theory of probability, he discovers that it is inadequate for the treatment of conditional probabilities, selections, and observations—all three of which are of prime importance in the modern theories of probability and statistics. In a number of recent formalizations of the theory of probability an additional operator (or logical constant) "if" has been introduced in order to handle conditional probabilities. Selections and observations were not introduced

into the formal schemes but were merely discussed as interpretations. In a previous paper the author showed that by a proper definition of "if" selections and observations could be formalized. However, this operator was defined in terms of certain extraneous concepts, namely, probabilities and order relations. In the present paper "if" is defined only by its relation to the Boolean operators, "and," "or," and "not." This more natural definition enables us to study the modified Boolean algebra per se and to investigate the structure of and the relations between its elements. (Received March 31, 1941.)

269. G. B. Dantzig: *Variance of the error of a mean computed by grouping.*

To compute the mean of a large number of observations, the range of the data is often divided into equal intervals and the mean is then computed as if each observation fell at the midpoint of the interval containing it. The error introduced by this short cut comes from two sources: (1) the size of the interval h and (2) the amount of off-centerness t, that is, the amount of shift required to bring the nearest mid-point to the origin. Several authors have noted that the average value over all t of the Sheppards' corrected moments (functions of h and the grouped moments) yields the true moments of the distribution. It is suggested here that the value of t should be selected at random, so that it is equally likely to have any value between 0 and h. Then the first grouped moment (now validly considered as a random variable) has for expected value the true first moment, and for variance $\frac{1}{2}h^2 \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x)p(y)B_2[(x-y)/h]dxdy$, where $p(x)$ is the density of distribution and $B_2(\theta)=\theta^2-\theta+1/6$ is periodic of period one (second Bernoulli polynomial). If the amount of off-centerness is randomly chosen, Sheppards' corrections are always applicable and the error approximable by square root of the variance. (Received March 10, 1941.)

270. G. E. Forsythe: *Cesàro summability of random variables.*

$\{X_k\}$ and $\{Y_k\}$ are sequences of independent real-valued random variables. $\{X_k\}$ is otherwise arbitrary, while $\{Y_k\}$ is a normal family (Lévy) with all $E(Y_k)=0$. $\|a_{nk}\|$ is the Cesàro C_α matrix ($0<\alpha<\infty$). Using theorems of Feller and Gnedenko, necessary and sufficient conditions are given for the existence of constants $\{d_{nk}\}$ so that both $\sum_{k=0}^{n} a_{nk}(X_k-d_{nk})$ and $a_{nk}(X_k-d_{nk})\to 0$ in probability as $n\to\infty$, uniformly in $k\leqq n$. If $\alpha\geqq 1$, d_{nk} may always be chosen independent of n and α. With natural definitions it then follows that for summability of $\{X_k\}$ to 0, the C_α methods get stronger as α increases. This increase in strength is non-trivial for $0<\alpha<1$, but is essentially trivial for $\alpha\geqq 1$. Using another Gnedenko theorem, a necessary and sufficient condition is given for C_α-summability of $\{Y_k\}$ to the Gaussian distribution $G(x)$. From this it is proved: if $0<\alpha<\frac{1}{2}$ or if $\sum_{k=0}^{\infty} E(Y_k^2)<\infty$, there can be no C_α-summability of $\{Y_k\}$ to $G(x)$; if $\frac{1}{2}\leqq\alpha<\beta$ and $1\leqq\beta$, $C_\alpha\subseteq C_\beta$; if $\frac{1}{2}\leqq\alpha<\beta$, there always exists $\{Y_k\}$ summable-C_α to $G(x)$ but not summable-C_β to $G(x)$, a perhaps surprising result. A close relation is shown between C_α-summability of $\{Y_k\}$ to $G(x)$ and summability of $\{E(Y_k^n)\}$ to $+\infty$ by ordinary Riesz means of order $2\alpha-2$. Nörlund summability to 0 of symmetric variables $\{Y_k\}$ is also treated. (Received March 31, 1941.)

271. J. A. Greenwood: *A theorem on probability assignments to events.*

Consider the mutually exclusive, exhaustive set of variates of a variable x with a probability distribution function $\phi(x)$ defined over $[a, b]$, and to each value of x an

associated number $P(x)$, $0 \leq P(x) \leq 1$. $P(x)$ may be called a probability assignment to the event x. Definition: $P(x)$ may be said to be a "sufficient" assignment if, for all x, $P(x) \geq LS\int_{D_x} d\phi(t)$ where $D_x = D_x(t \in [a, b]; P(t) \leq P(x))$. In order to ensure the existence of this integral and the succeeding one which defines $E[P(x)]$, it is assumed that D_x is a Borel set. The following theorem is then proved: A necessary condition that a probability assignment $P(x)$ to each of the mutually exclusive, exhaustive values of a stochastic variable x be "sufficient" is that $E[P(X)] = LS\int_a^b P(x)d\phi(x) \geq \frac{1}{2}$. (Received April 1, 1941.)

272. P. C. Hammer: *On fitting linear functions when all variables are subject to error.*

Among the many recent papers on fitting lines or planes in case all variates are subject to error is one by C. F. Roos which points out the lack of invariance of methods developed prior to that time. Among others the methods of Adcock, Pearson, and a generalization of Pearson's method by E. C. Rhodes seem to lack invariance. Assuming a general normal distribution of error and proceeding along classical lines this paper develops an implicit method of fitting linear functions in n variables which includes that of Pearson as a special case and which is invariant under coordinate transformation. A few geometrical interpretations are made and the cases not leading to a unique solution itemized. A comparison of this method with that of Roos is then made. (Received February 10, 1941.)

273. C. T. Hsu: *Two samples from normal bivariate populations.*

Two samples, each being of two variates (x_1, x_2) and (x_1', x_2'), of size n and n' respectively, are supposed to be drawn at random from two independent normal bivariate populations, with the following distributions: (1) $(2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2})^{-1}$ $\exp\{-1/(2(1-\rho^2))((x_1-\xi_1)/\sigma_1)^2 - 2\rho((x_1-\xi_1)/\sigma_1)((x_2-\xi_2)/\sigma_2) + ((x_2-\xi_2)/\sigma_2)^2]\}$, (2) $(2\pi\sigma_1'\sigma_2'(1-\rho'^2)^{1/2})^{-1}$ $\exp\{-1/(2(1-\rho'^2))[((x_1'-\xi_1')/\sigma_1')^2 - 2\rho'((x_1'-\xi_1')/\sigma_1')\cdot((x_2'-\xi_2')/\sigma_2') + (x_2'-\xi_2')/\sigma_2')^2]\}$, where $\xi_1, \xi_2, \sigma_1, \sigma_2, \rho; \xi_1', \xi_2', \sigma_1', \sigma_2', \rho'$ are the unknown parameters of the populations. Two hypotheses are considered concerning the comparison of correlation coefficients, namely, H_1: Assuming $\sigma_1 = \sigma_2$ and $\sigma_1' = \sigma_2'$; to test $\rho = \rho'$. H_2: Assuming $\sigma_1 = \sigma_2$, $\xi_1 = \xi_2$ and $\sigma_1' = \sigma_2'$, $\xi_1' = \xi_2'$; to test $\rho = \rho'$. Appropriate test criteria are derived for each hypothesis. The distribution of certain of the statistics are obtained in the special case where $n = n'$. Incidentally the distribution of the z of S. S. Wilks (Biometrika, vol. 24 (1932), p. 471) for $p-2$ and any values of a_1 and a_2 is studied. (Received April 1, 1941.)

274. Henry Scheffé: *Note on the reduction of χ^2 for fit of a frequency distribution.*

Pearson reduced χ^2 for the fit of an observed to a theoretical frequency distribution by the use of determinant theory and trigonometric identities (Philosophical Magazine, (5), vol. 50 (1900), pp. 160–163). The object of this note is to simplify the reduction by the use of matrix theory. (Received February 15, 1941.)

275. Abraham Wald: *On testing statistical hypotheses concerning several unknown parameters.*

Let $f(x, \theta_1, \cdots, \theta_k)$ be the probability density function of a random variable x involving k unknown parameters. For testing the simple hypothesis $\theta = \theta_1', \cdots, \theta_k = \theta_k'$, called hypothesis H, by means of n independent observations $x_1, \cdots, x_n$ on x we

choose a critical region W_n of the n-dimensional sample space and reject the hypothesis H if $E=(x_1, \cdots, x_n)$ falls inside W_n. Denote by $P(W_n \mid \theta_1, \cdots, \theta_k)$ the probability that E will fall in W_n under the assumption that $\theta_1, \cdots, \theta_k$ are the true values of the parameters and denote by $P_n(\theta_1, \cdots, \theta_k, \alpha)$ the least upper bound of $P(Z_n \mid \theta_1, \cdots, \theta_k)$ with respect to all regions Z_n for which $P(Z_n \mid \theta_1', \cdots, \theta_k') = \alpha$. A critical region W_n is called a most stringent test of the hypothesis H on the level of significance α if $P(W_n \mid \theta_1', \cdots, \theta_k') = \alpha$ and l.u.b. $\{P_n(\theta_1, \cdots, \theta_k, \alpha) - P(W_n \mid \theta_1, \cdots, \theta_k)\} \leqq$ l.u.b. $\{P_n(\theta_1, \cdots, \theta_k, \alpha) - P(Z_n \mid \theta_1, \cdots, \theta_k)\}$ (l.u.b. with respect to $\theta_1, \cdots, \theta_k$) for any region Z_n for which $P(Z_n \mid \theta_1', \cdots, \theta_k') = \alpha$. It is shown that the test of H based on the so-called likelihood ratio introduced by Neyman and Pearson is a most stringent test in the limit if $n \to \infty$. The foregoing definitions and results are extended also to testing composite hypotheses. (Received February 3, 1941.)

TOPOLOGY

276. G. E. Albert: *On separation spaces.* Preliminary report.

A. D. Wallace has introduced separation spaces (abstract 46-7-368), adopting as a primitive concept a binary relation $X \mid Y$, between pairs of non-vacuous subsets X and Y of an abstract set S. Subject to certain axioms, $X \mid Y$ can be used to define a topology in S which makes the resulting space completely equivalent to a T_1-topological space. A prominent part is played by the axiom: $X \mid Y$ implies $Y \mid X$. In the present paper separation spaces are studied in which this property of symmetry is discarded. It is shown that, subject to proper alterations of Wallace's axioms, separation spaces can be used to characterize T_0-topological spaces. The theory of asymmetrical separation is found to be particularly convenient for the definition of a topology in upper semi-continuous collections of type 2 (see R. L. Moore, Rice Institute Pamphlets, vol. 23, no. 1). Other applications of a more general nature are also indicated. (Received March 10, 1941.)

277. G. E. Albert and J. W. T. Youngs: *The structure of locally connected topological spaces.*

The present paper is a continuation of earlier work (abstract 46-3-138). Cyclic elements are defined and studied in the class L of spaces which satisfy the postulates: the space and the empty set are open; the intersection (sum) of finitely (arbitrarily) many open sets is open; the components of an open set are open. The hyperspace $\mathfrak{X}$ of all cyclic elements of any space $X \in L$ is topologized in such a way that: (1) $\mathfrak{X}$ is a space L, (2) $\mathfrak{X}$ is a strongly continuous image of X, and (3) $\mathfrak{X}$ is acyclic. A subclass H of L is called hereditary if $X \in H$ implies that the hyperspace of X is in H and every true cyclic element of X is a space in H. The class L is a hereditary class which contains the class P of all Peano spaces; however, it deviates widely from P. Hereditary subclasses of L are proposed which approximate more closely the class P. For example: one such class is composed of all spaces L which satisfy the T_0-separation axiom and which are strongly continuous images of the unit line interval. (Received March 10, 1941.)

278. Samuel Eilenberg: *Banach space methods in topology.* I.

For a given topological space X the Banach space $\mathfrak{X}$ of all continuous bounded real functions $\mathfrak{x}$ on X with norm l.u.b. $|\mathfrak{x}(x)|$ is considered. Banach has proved that two compact metric spaces, X_1 and X_2, are homeomorphic if and only if $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are iso-

metric. Stone (Transactions of this Society, vol. 41 (1937), p. 469) has generalized this theorem to normal bicompact spaces X. The author obtains the same result for normal compact spaces satisfying the first countability axiom. The method consists in an analysis of the convex subsets of the surface of the unit sphere in $\mathfrak{X}$. A complete reconstruction of X, under either Stone's or the author's hypotheses, may then be obtained. (Received March 12, 1941.)

279. Samuel Eilenberg: *Banach space methods in topology.* II.

A Banach space B is a direct product, $B_1 \times B_2$, of two of its closed linear subspaces B_1 and B_2, if every element $b \in B$ has a unique decomposition $b = b_1 + b_2$, $b_i \in B_i$, and if $\|b\| = \max(\|b_1\|, \|b_2\|)$. Given a topological space X decomposed into two disjoint closed sets X_1 and X_2, the Banach space $\mathfrak{X}$ (see abstract 47-5-278) is a direct product $\mathfrak{X}_1 \times \mathfrak{X}_2$ if $\mathfrak{X}_i$ is that subset of $\mathfrak{X}$ that consists of all functions x which vanish outside X_i. If X is normal, compact, and satisfies the first countability axiom, then the converse can be proved, that is, the above direct product decomposition is the most general one. In particular X is connected if and only if $\mathfrak{X}$ is indecomposable. (Received March 12, 1941.)

280. Samuel Eilenberg: *Irreducible transformations onto manifolds.*

A continuous mapping $f(X) = M$ of a metric compact space onto an n-dimensional manifold is called irreducible (equal to strongly irreducible in the terminology of G. T. Whyburn, American Journal of Mathematics, vol. 61 (1939), p. 820) if $M - f(A) \neq 0$ for every closed proper subset A of X. A point $y \in M$ is said to be covered essentially by f if there is a neighborhood U of y such that there is no mapping $g(X) \subset M - y$ such that: $g(x) = f(x)$ if $f(x) \in M - U$, $g(x) \in U$ if $f(x) \in U$. The following is proved: if f is irreducible and $y \in M$ is covered essentially, then $f^{-1}(y)$ is a continuum. In particular if f is irreducible and essential in the sense of Hopf, then f is monotone. (Received March 12, 1941.)

281. Harlan C. Miller: *Concerning compact unicoherent continua.*

In this paper it is proved that in order that the compact hereditarily decomposable continuum M be hereditarily unicoherent it is necessary and sufficient that M contain no continuum N such that N is a simple closed curve with respect to the elements of an upper semicontinuous collection of mutually exclusive continua filling up N. The author has previously proved that in order that the compact continuum M be atriodic and hereditarily unicoherent it is necessary and sufficient that of any three points of M there is one which weakly separates the other two from each other in M. It is now shown that (1) if M is a compact hereditarily decomposable continuum and there exists a positive integer k such that of any k points of M there is one which weakly separates two of the others from each other in M, then M is hereditarily unicoherent, and that the stipulation that M be hereditarily decomposable can be omitted only if $k < 5$, and (2) if M is a compact hereditarily decomposable continuum such that every subcontinuum of M is irreducible about some (closed) proper subset having only countably many components, then M is hereditarily unicoherent. (Received March 13, 1941.)

282. W. T. Puckett: *Concerning a transformation in the plane.* Preliminary report.

The single-valued continuous transformation $T(P) = P'$ will be called *quasi 0-regu-*

lar provided that (1) $T(P-Q)$ is interior for some discrete set Q and (2) for each $p \in P-Q$ and neighborhood $U(p)$ there exists a neighborhood $V(p)$ such that $(V(p)-Q)\cdot T^{-1}(x')$ lies in a single component of $(U(p)-Q)\cdot T^{-1}(x')$ for any $x' \in P'$. Under the hypotheses that P is the plane and each $T^{-1}(x')$ has only nondegenerate components it is shown that (1) every point of P' is a local separating point of order 2 and (2) any $T^{-1}(x')$ may be characterized as a discrete collection of open arcs, a finite number of which may have a point q of Q as a common end point. Moreover, the behavior of the transformation near such a point q is exhibited. This theory will be applied to a topological characterization of potential functions. (Received March 10, 1941.)

283. G. E. Schweigert: *Border transformations.* II.

If A is compact metric and if the light factor of the monotone light factorization of a continuous transformation, $T(A)=B$, is interior, then T is said to be quasi-monotone. This new definition (A. D. Wallace, Duke Mathematical Journal, vol. 7 (1940)) gives a desirable property which, an example shows, does not hold under the old except for special spaces. Quasi-monotone is characterized by the statement: a sequence converging to $y \in B$ implies that the limit superior of the counterimages of the sequence intersects each component of $T^{-1}(y)$. This permits comparison with the border transformation where the limit of the counterimages is the border of $T^{-1}(y)$. Also, if $I(M)$ denotes the interior of M, the necessary condition $T^{-1}(I(Y))+\sum I(T^{-1}(y))=I(T^{-1}(Y))$, where the summation is over the border of Y, is shown to be sufficient that T be a border transformation. Two other necessary conditions are shown to be sufficient. (Received March 13, 1941.)

284. M. E. Shanks: *The space of all metrics on a compact space.*

If X is a compact metrizable space, the space $M(X)$ of all metrics on X, which preserve the topology, and their limits is a complete, separable positively-linear normed space. $M(X)$ is unique, that is, $M(X)$ is congruent to $M(Y)$ if and only if X is homeomorphic to Y. $M(X)$ contains in a sense all continuous transformations of X. If $M^*(X)$ denotes the linear extension of $M(X)$, then $M^*(X)$ is a universal metric space if X has the power of the continuum. In fact it is possible to define a whole class of universal metric spaces by taking different classes of metrices on X. The methods are largely those of Banach. (Received March 14, 1941.)

285. N. E. Steenrod and A. W. Tucker: *Real n-quadrics as sphere-bundles.*

By a suitable choice of homogeneous coordinates in real projective $(n+1)$-space the equation of a real n-quadric Q can be written $x_0^2+x_1^2+\cdots+x_k^2=x_{k+1}^2+\cdots+x_{n+1}^2$ $(k \leqq n-k)$. The mapping $(x_0:x_1:\cdots:x_{n+1})\to(x_0:x_1:\cdots:x_k:0:\cdots:0)$ sections the quadric Q into a continuous collection of $(n-k)$-spheres, one for each point of projective k-space. This collection is a sphere-bundle in the sense of Whitney (Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 148–153). Using results of Radon on linear sets of orthogonal matrices (Abhandlungen des mathematischen Seminars der Hamburgischen Universität, vol. 1 (1921), pp. 1–14) it can be shown that Q is topologically equivalent to the product $P\times S$ of projective k-space P and an $(n-k)$-sphere S provided $n-k$ is odd and k is small enough (if λ denotes the exponent of the highest power of 2 which divides $n-k+1$ then, according as $\lambda \equiv 0, 1, 2, 3 \bmod 4$,

$k<2\lambda+1$, 2λ, 2λ, $2\lambda+2$, respectively). For $n-k$ odd and k larger it seems probable that Q and $P\times S$ are not topologically equivalent, although they have isomorphic homology groups. For $n-k$ even and $k>0$ the quadric Q and the product $P\times S$ differ in orientability but have isomorphic homology groups mod 2. (Received March 14, 1941.)

286. A. W. Tucker: *Simplicial bands.*

Consider a simplicial complex formed by a circular sequence of k n-simplexes, each of which has an $n-1$-face in common with the next ($k\geqq 2n+1$, necessarily). This *simplicial band* has as many p-simplexes as $n-p$-simplexes and its boundary is a regular $n-1$-polyhedron. If n and k are both even or if n is odd, it is the topological product of a circle and a closed $n-1$-cell. If n is even and k odd, it is an n-dimensional analogue of a Möbius band, obtained from an n-cylinder by a sense-reversing identification of the base and cap. If n or k is odd, it is topologically equivalent to each of the following: (1) the symmetric product of n circles; (2) the closed region of real projective n-space which represents the subset of real equations of degree less than or equal to n with real roots, when coefficients are used as homogeneous coordinates; and (3) the quadric-bounded projective region $x_0^2+x_1^2\geqq x_2^2+\cdots+x_n^2$. (Received March 8, 1941.)

287. A. D. Wallace: *A fixed-point theorem for trees.*

Let T be a compact Hausdorff space which is locally connected and which is acyclic in the sense that every finite open covering admits a finite open refinement whose nerve is a tree. Let q be a function which assigns to each point t in T a continuum qt in T, and which is continuous in the sense that if t and a neighborhood U of qt are given then there is a neighborhood V of t such that if t' is in V then qt' is in U. In this note it is shown that *there is a t_0 in T such that t_0 is in qt_0.* This is a generalization of the Scherrer fixed-point theorem for acyclic continuous curves and is analogous to a recent result of Kakutani. As a corollary to our result we deduce a known result on the nonexistence of free monotone transformations into trees and a new theorem on the existence of coincidences of monotone transformations into trees. (Received March 7, 1941.)

288. P. A. White: *A decomposition of true cyclic elements by means of continua.*

In this paper a decomposition of a cyclicly connected locally connected continuum is defined by dividing the continuum into certain sets which have the property that no two points in them can be separated by a continuum. These sets are divided into two types according as they do or do not contain points of Menger order two. It is shown that each set of the first type is contained in a simple closed curve and that they are at most countable in number, while each set of the second type is at most countable. Other results are given concerning the components of these sets and of their complements. (Received March 31, 1941.)

289. G. W. Whitehead: *Homotopy properties of rotation groups.*

This paper continues the investigation of homotopy properties of the rotation groups R_n of n-spheres begun by Hurewicz and Steenrod (Proceedings of the National

Academy of Sciences, vol. 27 (1941), pp. 61–64). The homotopy groups $\Pi_i(R_n)$ are computed for $i=3, 4, 5$ and all n, with the following results: $\Pi_3(R_n)$ is the infinite cyclic group ($n=2$, $n\geqq 4$); $\Pi_3(R_3)$ is the direct sum of two such groups; $\Pi_4(R_2)=\Pi_4(R_4)$ is the cyclic group of period 2, $\Pi_4(R_3)$ is the direct sum of two such groups, $\Pi_4(R_n)=0$ ($n\geqq 5$); $\Pi_5(R_n)=0$ ($n\neq 5$), and $\Pi_5(R_5)$ is the infinite cyclic group. It is proved that if $n>1$ and $n\not\equiv 3 \pmod 4$, there exists no map f of S^n into R_n such that the projection Πf into S^n has odd degree. Thus for these values of n there do not exist n independent fields of tangent vectors on S^n, and the tangent sphere-bundle is not simple. In fact any two vector fields are somewhere dependent. (Received March 13, 1941.)

290. G. T. Whyburn: *Almost periodicity.*

Let $f(X)=X$ be a homeomorphism where X is a metric space. It is shown that f is pointwise periodic if and only if for any subset Y of X, $f(Y)\subset Y$ implies $f(Y)=Y$ and that, for compact sets X, f is pointwise almost periodic (see Ayres, Fundamenta Mathematicae, vol. 33, reprint) if and only if $f(K)\subset K$ implies $f(K)=K$ for every compact subset K of X. A lemma is then proven to the effect that if X is a continuum and $X=H+K$ is any division of X into continua H and K such that $H\cdot K$ is a single point p and $H\cdot f(H)\neq 0\neq K\cdot f(K)$, then pointwise almost periodicity of f at p implies $f(p)=p$. On the basis of this characterization and lemma, a considerably simplified development (using nowhere the notion of linear order) is given of the results of Ayres on pointwise almost periodic homeomorphisms—at the same time extending them to semi-locally-connected continua. (Received March 12, 1941.)

291. G. T. Whyburn: *Orbit decompositions.*

Let $f(X)\subset X$ be continuous where X is metric. For any $x\in X$, the set A_x of all $y\in X$ such that $f^m(x)=f^n(y)$ for positive integers m and n will be called the *orbit* of x. A subset A of X is *completely invariant* provided $f(A)=A$ and $f^{-1}(A)=A$. The set A_x then turns out to be the smallest completely invariant subset of X containing x. The decomposition of X into the (disjoint) sets $[A_x]$ is called the *orbit decomposition*. It may be obtained also by means of the equivalence relation: $x\equiv y$ provided $f^m(x)=f^n(y)$ for some $m, n>0$. Assuming X compact and the orbits closed, it is found that the orbit decomposition is upper semi-continuous provided f has *equi-continuous powers*, that is, for any $x\in X$ and $\epsilon>0$ a neighborhood U of x exists so that $\rho[f^n(x), f^n(y)]<\epsilon$ for all $y\in U$ and all $n>0$. If f is pointwise periodic, the converse holds and further, in this case, the condition is equivalent to *regular almost periodicity* of f, that is, for any $\epsilon>0$ an n exists so that $\rho[x, f^{mn}(x)]<\epsilon$ for all $x\in X$ and all $m>0$. Also, interiority of f implies pointwise periodicity, and the orbit decomposition is actually continuous. (Received March 12, 1941.)

292. G. T. Whyburn: *Regular almost periodicity.*

A homeomorphism $f(X)=X$ is regularly almost periodic if for each $\epsilon>0$ an integer n exists such that $\rho[x, f^{mn}(x)]<\epsilon$ for all $x\in X$ and all integers $m>0$. Regular almost periodicity is cyclicly extensible in semi-locally-connected continua X. If each cyclic element of such an X is locally euclidean, any pointwise periodic mapping on the non-end-points of X is regularly almost periodic on X. A simplified treatment of the results of Hall and Schweigert (Duke Mathematical Journal, vol. 4 (1938), p. 719; this Bulletin, vol. 46 (1940), p. 963) is given making no use of transfinite ordinals or induction. It is shown that any pointwise periodic mapping $f(X)\subset X$, X compact, is monotoni-

cally equivalent to a pointwise periodic and regularly almost periodic mapping $g(X')=X'$, that is, there exists a monotone mapping $\phi(X)=X'$ such that $g(x')=\phi f\phi^{-1}(x')$, $x'\in X'$. The author obtains also a new proof of the 2-dimensional case of a theorem of Montgomery (American Journal of Mathematics, vol. 59 (1937), p. 118) based on the fact that a finite-to-one interior mapping of any 2-manifold (compact or not) into a connected set has a finite degree k and the set of points of multiplicity less than k is one-dimensional, rather than on Newman's theorem. (Received March 25, 1941.)

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